Bimetric equations in standard 3+1 form in spherical symmetry

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Evolution equations

Evolution equations for the spatial metrics

$$\begin{aligned}
\partial_t A &= \overset{(K)}{-}\,\alpha A K_1 + \partial_r(qA + \alpha v),\\
\partial_t \widetilde{A} &= \overset{(K)}{-}\,\widetilde{\alpha}\widetilde{A}\widetilde{K}_1 + \partial_r(q\widetilde{A} - \widetilde{\alpha} v),\\
\partial_t B &= \overset{(K)}{-}\,\alpha B K_2 + \big(q + \alpha A^{-1} v\big)\partial_r B,\\
\partial_t \widetilde{B} &= \overset{(K)}{-}\,\widetilde{\alpha}\widetilde{B}\widetilde{K}_2 + \big(q - \widetilde{\alpha}\widetilde{A}^{-1} v\big)\partial_r \widetilde{B}.
\end{aligned}$$

Evolution equations for the extrinsic curvatures

$$\begin{aligned}
\partial_t K_1 &= \big(q + \alpha A^{-1} v\big)\partial_r K_1 \overset{(K)}{+} \alpha K_1\big(K_1 + 2K_2\big) \overset{(K)}{-} \alpha\kappa_g\Big\{ J_1 - \frac{1}{2}(J - \rho) \Big\}\\
&\quad \overset{(K)}{-}\left(\frac{\partial_r\alpha}{A^2}\frac{\partial_r A}{A} - \frac{\partial_r^2\alpha}{A^2} + 2\frac{\alpha}{A^2}\frac{\partial_r A}{A}\frac{\partial_r B}{B} - 2\frac{\alpha}{A^2}\frac{\partial_r^2 B}{B}\right),\\
\partial_t \widetilde{K}_1 &= \big(q - \widetilde{\alpha}\widetilde{A}^{-1} v\big)\partial_r \widetilde{K}_1 \overset{(K)}{+} \widetilde{\alpha}\widetilde{K}_1\big(\widetilde{K}_1 + 2\widetilde{K}_2\big) \overset{(K)}{-} \widetilde{\alpha}\kappa_f\Big\{ \widetilde{J}_1 - \frac{1}{2}(\widetilde{J} - \widetilde{\rho}) \Big\}\\
&\quad \overset{(K)}{-}\left(\frac{\partial_r\widetilde{\alpha}}{\widetilde{A}^2}\frac{\partial_r \widetilde{A}}{\widetilde{A}} - \frac{\partial_r^2\widetilde{\alpha}}{\widetilde{A}^2} + 2\frac{\widetilde{\alpha}}{\widetilde{A}^2}\frac{\partial_r \widetilde{A}}{\widetilde{A}}\frac{\partial_r \widetilde{B}}{\widetilde{B}} - 2\frac{\widetilde{\alpha}}{\widetilde{A}^2}\frac{\partial_r^2 \widetilde{B}}{\widetilde{B}}\right),\\
\partial_t K_2 &= \big(q + \alpha A^{-1} v\big)\partial_r K_2 \overset{(K)}{+} \alpha K_2\big(K_1 + 2K_2\big) \overset{(K)}{-} \alpha\kappa_g\Big\{ J_2 - \frac{1}{2}(J - \rho) \Big\}\\
&\quad \overset{(K)}{-}\left(\frac{\alpha}{B^2} - \frac{\partial_r\alpha}{A^2}\frac{\partial_r B}{B} + \frac{\alpha}{A^2}\frac{\partial_r A}{A}\frac{\partial_r B}{B} - \frac{\alpha}{A^2}\frac{(\partial_r B)^2}{B^2} - \frac{\alpha}{A^2}\frac{\partial_r^2 B}{B}\right),\\
\partial_t \widetilde{K}_2 &= \big(q - \widetilde{\alpha}\widetilde{A}^{-1} v\big)\partial_r \widetilde{K}_2 \overset{(K)}{+} \widetilde{\alpha}\widetilde{K}_2\big(\widetilde{K}_1 + 2\widetilde{K}_2\big) \overset{(K)}{-} \widetilde{\alpha}\kappa_f\Big\{ \widetilde{J}_2 - \frac{1}{2}(\widetilde{J} - \widetilde{\rho}) \Big\}\\
&\quad \overset{(K)}{-}\left(\frac{\widetilde{\alpha}}{\widetilde{B}^2} - \frac{\partial_r\widetilde{\alpha}}{\widetilde{A}^2}\frac{\partial_r \widetilde{B}}{\widetilde{B}} + \frac{\widetilde{\alpha}}{\widetilde{A}^2}\frac{\partial_r \widetilde{A}}{\widetilde{A}}\frac{\partial_r \widetilde{B}}{\widetilde{B}} - \frac{\widetilde{\alpha}}{\widetilde{A}^2}\frac{(\partial_r \widetilde{B})^2}{\widetilde{B}^2} - \frac{\widetilde{\alpha}}{\widetilde{A}^2}\frac{\partial_r^2 \widetilde{B}}{\widetilde{B}}\right).
\end{aligned}$$

To be solved at t_0

Scalar constraints

$$\begin{aligned}
(2K_1 + K_2)K_2 + \frac{1}{A^2}\left(\frac{A^2}{B^2} + 2\frac{\partial_r A}{A}\frac{\partial_r B}{B} - \frac{(\partial_r B)^2}{B^2} - 2\frac{\partial_r^2 B}{B}\right) &= \kappa_g \rho,\\
(2\widetilde{K}_1 + \widetilde{K}_2)\widetilde{K}_2 + \frac{1}{\widetilde{A}^2}\left(\frac{\widetilde{A}^2}{\widetilde{B}^2} + 2\frac{\partial_r \widetilde{A}}{\widetilde{A}}\frac{\partial_r \widetilde{B}}{\widetilde{B}} - \frac{(\partial_r \widetilde{B})^2}{\widetilde{B}^2} - 2\frac{\partial_r^2 \widetilde{B}}{\widetilde{B}}\right) &= \kappa_f \widetilde{\rho}.
\end{aligned}$$

Vector constraints

$$\begin{aligned}
2\Big\{(K_1 - K_2)\frac{\partial_r B}{B} - \partial_r K_2\Big\} &= \overset{(K)}{+}\,\kappa_g j_r,\\
2\Big\{(\widetilde{K}_1 - \widetilde{K}_2)\frac{\partial_r \widetilde{B}}{\widetilde{B}} - \partial_r \widetilde{K}_2\Big\} &= \overset{(K)}{+}\,\kappa_f \widetilde{j}_r.
\end{aligned}$$

Combined vector constraints

$$\begin{aligned}
&\kappa_f \widetilde{A} B\big(K_1\partial_r B - K_2\partial_r B - B\partial_r K_2\big)\\
&\quad + \kappa_g A\widetilde{B}\big(\widetilde{K}_1\partial_r\widetilde{B} - \widetilde{K}_2\partial_r\widetilde{B} - \widetilde{B}\partial_r\widetilde{K}_2\big) = 0.
\end{aligned}$$

The radial shift separation

$$p = \overset{(K)}{-}\,\frac{2}{\kappa_g \widetilde{A} B \langle R\rangle^2_1}\big(K_1\partial_r B - K_2\partial_r B - B\partial_r K_2\big).$$

Bimetric conservation law ('secondary constraint')

$$\begin{aligned}
&\widetilde{A}\Big(\widetilde{K}_1\langle R\rangle^2_1 + 2\widetilde{K}_2 R\langle R\rangle^1_2\Big) - A\Big(K_1\langle R\rangle^2_1 + 2K_2\langle R\rangle^1_1\Big)\\
&\quad + 2A\widetilde{K}_2\lambda R\langle R\rangle^1_1 - 2\widetilde{A}K_2\lambda\langle R\rangle^1_2\\
&\quad \overset{(K)}{+} 2p\left(\langle R\rangle^1_1\frac{A}{\widetilde{A}}\frac{\partial_r\widetilde{B}}{B} + \langle R\rangle^1_2\frac{\widetilde{A}}{A}\frac{\partial_r B}{B}\right) \overset{(K)}{+} \lambda^{-1}\langle R\rangle^2_1\partial_r p = 0.
\end{aligned}$$

Relation between two lapses

Definitions

Metric

$$\begin{aligned}
g &= -\alpha^2\mathrm{d}t^2 + A^2(\mathrm{d}r + \beta\,\mathrm{d}t)^2 + B^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2),\\
f &= -\widetilde{\alpha}^2\mathrm{d}t^2 + \widetilde{A}^2(\mathrm{d}r + \widetilde{\beta}\,\mathrm{d}t)^2 + \widetilde{B}^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).
\end{aligned}$$

Projections of V_g

$$\rho = -\left[\langle R\rangle^2_0 + \lambda\frac{\widetilde{A}}{A}\langle R\rangle^2_1\right], \qquad j_r = -p\widetilde{A}\langle R\rangle^2_1,$$

$$J_1 = \langle R\rangle^2_0 + \left[\frac{1}{\lambda}\left(\frac{\widetilde{\alpha}}{\alpha} + \frac{\widetilde{A}}{A}\right) - \lambda\frac{\widetilde{A}}{A}\right]\langle R\rangle^2_1,$$

$$J_2 = \langle R\rangle^1_0 + \frac{\widetilde{\alpha}\widetilde{A}}{\alpha A}\langle R\rangle^2_1 + \frac{1}{\lambda}\left(\frac{\widetilde{\alpha}}{\alpha} + \frac{\widetilde{A}}{A}\right)\langle R\rangle^1_1.$$

Projections of V_f

$$\widetilde{\rho} = -\left[\langle R\rangle^2_2 + \lambda\frac{\widetilde{A}}{A}\langle R\rangle^2_1\right]\frac{1}{R^2}, \qquad \widetilde{j}_r = pA\langle R\rangle^2_1\frac{1}{R^2},$$

$$\widetilde{J}_1 = \left\{\langle R\rangle^2_2 + \left[\frac{1}{\lambda}\left(\frac{\alpha}{\widetilde{\alpha}} + \frac{A}{\widetilde{A}}\right) - \lambda\frac{A}{\widetilde{A}}\right]\langle R\rangle^2_1\right\}\frac{1}{R^2},$$

$$\widetilde{J}_2 = \left\{\langle R\rangle^1_3 + \frac{\alpha A}{\widetilde{\alpha}\widetilde{A}}\langle R\rangle^1_1 + \frac{1}{\lambda}\left(\frac{\alpha}{\widetilde{\alpha}} + \frac{A}{\widetilde{A}}\right)\langle R\rangle^1_2\right\}\frac{1}{R}.$$

The partial span, $R \coloneqq \widetilde{B}/B$

$$\langle R\rangle^n_k \coloneqq \overset{(V)}{-}\,m^4\sum_{i=0}^{n}\binom{n}{i}\beta_{i+k}R^i,$$

$$\langle R\rangle^n_k = \langle R\rangle^{n-1}_k + R\langle R\rangle^{n-1}_{k+1}, \quad \langle R\rangle^0_k = \overset{(V)}{-}\,m^4\beta_k.$$

Variables

g sector: A, B, K_1, K_2, f sector: $\widetilde{A}, \widetilde{B}, \widetilde{K}_1, \widetilde{K}_2$,
Gauge: α, q (also $\widetilde{\alpha}$).

The radial shift separation

$$\beta = q + \alpha A^{-1}p\lambda^{-1}, \qquad \widetilde{\beta} = q - \widetilde{\alpha}\widetilde{A}^{-1}p\lambda^{-1}$$

$$v = p\lambda^{-1}, \qquad \lambda^2 = 1 + p^2.$$