

Path Integrals: Perturbation expansion for the ϕ^4 -theory

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Lorentzian Path Integral

$$\int d^4x \equiv \int dx^0 \dots \int dx^3, \quad \int d^4p \equiv \int dp^0 \dots \int dp^3,$$

$$\square \equiv \partial_0^2 - \nabla^2 = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2,$$

$$x^2 \equiv (x^0)^2 - \mathbf{x}^2, \quad p^2 \equiv (p^0)^2 - \mathbf{p}^2, \quad p \cdot x \equiv p^0 x^0 - \mathbf{p} \cdot \mathbf{x}$$

The Wick Rotation $iS_L = -S_E$

$$x^0 \equiv -ix^4, \quad x^4 \equiv ix^0, \quad p^0 \equiv ip^4, \quad p^4 \equiv -ip^0,$$

$$\partial_0 \equiv i\partial_4, \quad dx^0 \equiv -id^4x^4, \quad \square_L \equiv -\square_E,$$

$$\int d^4x_L \equiv -i \int d^4x_E, \quad \int d^4p_L \equiv i \int d^4p_E,$$

$$x_L^2 \equiv -x_E^2, \quad p_L^2 \equiv -p_E^2,$$

$$p^0 x^0 = p^4 x^4, \quad (p \cdot x)_L \neq (p \cdot x)_E,$$

$$\delta(x^0) \text{ using FT def.} = i\delta(x^4).$$

Euclidean Path Integral

$$\int d^4x \equiv \int dx^1 \dots \int dx^4, \quad \int d^4p \equiv \int dp^1 \dots \int dp^4,$$

$$\square \equiv \partial_4^2 + \nabla^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 + \partial_4^2,$$

$$x^2 \equiv \mathbf{x}^2 + (x^4)^2, \quad p^2 \equiv \mathbf{p}^2 + (p^4)^2, \quad p \cdot x \equiv \mathbf{p} \cdot \mathbf{x} + p^4 x^4$$

Action

$$\mathcal{L}[\phi] = \mathcal{L}_0[\phi] + \mathcal{L}_{\text{int}}[\phi], \quad \mathcal{L}_0[\phi] = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}m^2 \phi^2, \quad \mathcal{L}_{\text{int}}[\phi] = \frac{\lambda}{4!} \phi^4,$$

$$S[\phi] = S_0[\phi] + S_{\text{int}}[\phi], \quad S_0[\phi] = \int d^4x \mathcal{L}_0[\phi], \quad S_{\text{int}}[\phi] = \int d^4x \mathcal{L}_{\text{int}}[\phi].$$

Feynman propagator

$$\Delta_F^E(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x_1 - x_2)} \frac{1}{p^2 + m^2}$$

$$\Delta_F^E(x_1 - x_2) = \langle 0 | T \{ \hat{\phi}(x_1), \hat{\phi}(x_2) \} | 0 \rangle$$

$$\Delta_F^E(x_1 - x_2) = i\Delta_F(x_1 - x_2)$$

The euclidean causal propagator does not require the $i\epsilon$ -prescription!

Generating Functional

$$Z_0[J] = \int \mathcal{D}\phi \exp\left(-S_0 + \int d^4z J(z)\phi(z)\right)$$

$$Z[J] = \int \mathcal{D}\phi \exp\left(\boxed{-S_{\text{int}}} - S_0 + \int d^4z J(x)\phi(z)\right)$$

n -point correlation function

$$G_E^{(n)}(x_1, \dots, x_n) = \left(\frac{1}{Z[J]} \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} Z[J] \right)_{J=0}$$

Action

$$\mathcal{L}[\phi] = \mathcal{L}_0[\phi] + \mathcal{L}_{\text{int}}[\phi], \quad \mathcal{L}_0[\phi] = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2 \phi^2, \quad \mathcal{L}_{\text{int}}[\phi] = -\frac{\lambda}{4!} \phi^4,$$

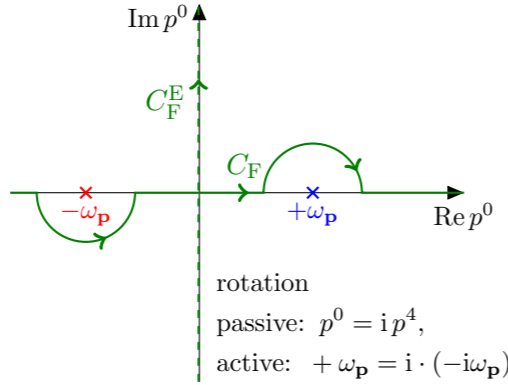
$$S[\phi] = S_0[\phi] + S_{\text{int}}[\phi], \quad S_0[\phi] = \int d^4x \mathcal{L}_0[\phi], \quad S_{\text{int}}[\phi] = \int d^4x \mathcal{L}_{\text{int}}[\phi].$$

Feynman propagator

$$\Delta_F(x_1 - x_2) = \int_{C_F} \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x_1 - x_2)} \frac{1}{p^2 - m^2}$$

$$i\Delta_F(x_1 - x_2) = \langle 0 | T \{ \hat{\phi}(x_1), \hat{\phi}(x_2) \} | 0 \rangle$$

$$i\Delta_F(x_1 - x_2) = \Delta_F^E(x_1 - x_2)$$

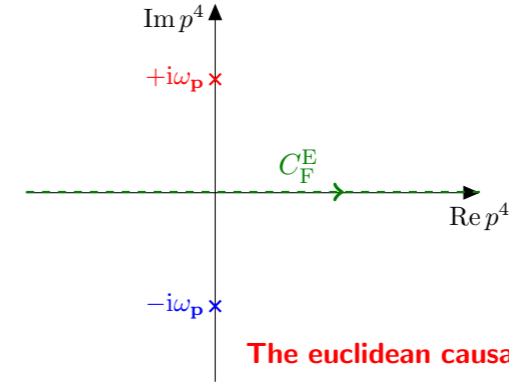


Note: The signs are chosen such that the product of time and energy, $x^0 p^0 = x^4 p^4$, is invariant. This is an arbitrary choice which ensures that the travelling direction of a plane wave remains unchanged when the coordinates are transformed.

For an arbitrary $f(p^2)$, it holds:

$$\int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} f(p^2) = i \int \frac{d^4p_E}{(2\pi)^4} e^{-ip_E \cdot x_E} f(-p_E^2)$$

Since $p \cdot x \neq p_E \cdot x_E$, the above relation requires the insensitivity under spatial reflections: $\mathbf{p} \rightarrow -\mathbf{p}$.



Generating Functional

$$Z_0[J] = \int \mathcal{D}\phi \exp\left(iS_0 + i \int d^4z J(z)\phi(z)\right)$$

$$Z[J] = \int \mathcal{D}\phi \exp\left(\boxed{iS_{\text{int}}} + iS_0 + i \int d^4z J(x)\phi(z)\right)$$

$$Z_0[J] = Z_0[0] \exp\left(-\frac{i}{2} \int d^4z_1 d^4z_2 J(z_1) \Delta_F(z_1 - z_2) J(z_2)\right)$$

$$Z[J] = \exp\left(\boxed{iS_{\text{int}}}\left[\frac{\delta}{i\delta J(y)}\right]\right) Z_0[J]$$

$$Z_0[0] = \int \mathcal{D}\phi \exp(iS_0) = (\det(\square + m^2))^{-1/2}$$

$$Z_0[J] = Z_0[0] \exp\left(\frac{1}{2} \int d^4z_1 d^4z_2 J(z_1) \Delta_F^E(z_1 - z_2) J(z_2)\right)$$

$$Z[J] = \exp\left(\boxed{-S_{\text{int}}}\left[\frac{\delta}{\delta J(y)}\right]\right) Z_0[J]$$

$$Z_0[0] = \int \mathcal{D}\phi \exp(-S_0) = (\det(-\square + m^2))^{-1/2}$$

n -point correlation function

$$G^{(n)}(x_1, \dots, x_n) = \left(\frac{1}{Z[J]} \frac{\delta}{i\delta J(x_1)} \dots \frac{\delta}{i\delta J(x_n)} Z[J] \right)_{J=0}$$

$$G^{(n)}(x_1, \dots, x_n) = G_E^{(n)}(x_1, \dots, x_n),$$

$$G_0^{(2)}(x_1, x_2) = i\Delta_F(x_1 - x_2) = \Delta_F^E(x_1 - x_2) = G_{0,E}^{(2)}(x_1, x_2)$$

First-order expansion of the 2-point correlation function $G^{(2)}(x_1, x_2)$ for the ϕ^4 -theory in the euclidean framework: (for E \rightarrow L, replace $-\lambda \rightarrow i\lambda$)

$$= \frac{\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} Z[J]}{Z[J]} \Bigg|_{J=0} = \frac{\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \exp\left(\boxed{-S_{\text{int}}}\left[\frac{\delta}{\delta J(y)}\right]\right) Z_0[J]}{\exp\left(\boxed{-S_{\text{int}}}\left[\frac{\delta}{\delta J(y)}\right]\right) Z_0[J]} \Bigg|_{J=0} = \frac{\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \exp\left(-\frac{\lambda}{4!} \int d^4y \frac{\delta^4}{\delta J(y)^4}\right) [Z_0[0] \exp\left(\frac{1}{2} \int d^4z_1 d^4z_2 J(z_1) \Delta_F^E(z_1 - z_2) J(z_2)\right)]}{\exp\left(-\frac{\lambda}{4!} \int d^4y \frac{\delta^4}{\delta J(y)^4}\right) [Z_0[0] \exp\left(\frac{1}{2} \int d^4z_1 d^4z_2 J(z_1) \Delta_F^E(z_1 - z_2) J(z_2)\right)]} \Bigg|_{J=0}$$

Now, Taylor expand the exponential function for the interaction to the required order in λ . Also, Taylor expand the exponential functions with the source to match the number of functional derivatives. Note that $J = 0$ will cancel all unused source terms after the expansion (so they can be ignored).

$$= \frac{\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \left(1 - \frac{\lambda}{4!} \int d^4y \frac{\delta^4}{\delta J(y)^4} + \mathcal{O}(\lambda^2)\right) \left[(\text{ign. } \frac{1}{0!}) + \frac{1}{1!} \left(\frac{1}{2}\right)^1 \int d^4z_1 d^4z_2 J(z_1) J(z_2) \Delta_F^E(z_1 - z_2) + (\text{ign. } \frac{1}{2!}) + \frac{1}{3!} \left(\frac{1}{2}\right)^3 \int d^4z_1 \dots d^4z_6 J(z_1) \dots J(z_6) \Delta_F^E(z_1 - z_2) \Delta_F^E(z_3 - z_4) \Delta_F^E(z_5 - z_6) \right]}{\left(1 - \frac{\lambda}{4!} \int d^4y \frac{\delta^4}{\delta J(y)^4} + \mathcal{O}(\lambda^2)\right) \left[1 + (\text{ign. } \frac{1}{1!}) + \frac{1}{2!} \left(\frac{1}{2}\right)^2 \int d^4z_1 \dots d^4z_4 J(z_1) \dots J(z_4) \Delta_F^E(z_1 - z_2) \Delta_F^E(z_3 - z_4)\right]} \Bigg|_{J=0}$$

$$= \frac{\frac{1}{1!} \left(\frac{1}{2}\right)^1 \times 2 \times \Delta_F^E(x_1 - x_2) + \frac{1}{3!} \left(\frac{1}{2}\right)^3 \times 6 \times 4! \times \left(-\frac{\lambda}{4!} \int d^4y\right) \Delta_F^E(x_1 - x_2) \Delta_F^E(y - y) \Delta_F^E(y - y) + \frac{1}{3!} \left(\frac{1}{2}\right)^3 \times 6 \times 4 \times 4! \times \left(-\frac{\lambda}{4!} \int d^4y\right) \Delta_F^E(x_1 - y) \Delta_F^E(y - x_2) \Delta_F^E(y - y) + \mathcal{O}(\lambda^2)}{1 + \frac{1}{2!} \left(\frac{1}{2}\right)^2 \times 4! \times \left(-\frac{\lambda}{4!} \int d^4y\right) \Delta_F^E(y - y) \Delta_F^E(y - y) + \mathcal{O}(\lambda^2)}$$

$$= \frac{\Delta_F^E(x_1 - x_2) + 3 \times \Delta_F^E(x_1 - x_2) \left(-\frac{\lambda}{4!} \int d^4y\right) \Delta_F^E(y - y) \Delta_F^E(y - y) + 12 \times \left(-\frac{\lambda}{4!} \int d^4y\right) \Delta_F^E(x_1 - y) \Delta_F^E(y - x_2) \Delta_F^E(y - y) + \mathcal{O}(\lambda^2)}{1 + 3 \times \left(-\frac{\lambda}{4!} \int d^4y\right) \Delta_F^E(y - y) \Delta_F^E(y - y) + \mathcal{O}(\lambda^2)} = \Delta_F^E(x_1 - x_2) - \frac{\lambda}{2} \int d^4y \Delta_F^E(x_1 - y) \Delta_F^E(y - x_2) \Delta_F^E(y - y) + \mathcal{O}(\lambda^2),$$

$$\implies G^E(p) = G_0^E(p) - \frac{\lambda}{2} G_0^E(p)^2 \Delta_F^E(0) + \mathcal{O}(\lambda^2) = \frac{1}{p^2 + m^2} \left(1 - \frac{\lambda}{2} \frac{1}{p^2 + m^2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2}\right) + \mathcal{O}(\lambda^2) = \frac{1}{p^2 + m^2 + \frac{\lambda}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2}} + \mathcal{O}(\lambda^2), \quad \text{with hard cut-off: } \delta m^2 = \frac{\lambda}{2} \int_0^\Lambda \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} = \frac{\lambda}{32\pi^2} \left(\Lambda^2 - m^2 \log \frac{\Lambda^2 + m^2}{m^2}\right).$$

Note: $\frac{\delta^n}{\delta J(y)^n} J(z_1) \dots J(z_n) = \sum_\sigma \delta(z_{\sigma(1)} - y) \dots \delta(z_{\sigma(n)} - y)$, where σ runs over $n!$ permutations.

Here, the disconnected loops in the denominator cancel out. The general result is the linked-cluster theorem which states that the vacuum diagrams cancel exactly out to all orders in perturbation theory. (The theorem is valid for all the n -point functions and for any local theory.)