Tutorial 1

Topics for today

- Special relativity and Lorentz invariance
- Index structure and index gymnastics; the form of a Lagrangian
- Notation, 2π and signs; Fourier transform, Dirac delta function

Oncoming:

- Natural Units; Ladder operators, Heisenberg algebra; Gamma matrices, Dirac algebra...

Problem 1. (a) Show that for *any* Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ it always hold,

$$\det(\Lambda^{\mu}{}_{\nu}) = \pm 1 \quad \text{and} \quad \left|\Lambda^{0}{}_{0}\right| \geq 1. \tag{0.1}$$

(b) (Schwartz, 2.4) Is the transformation $Y : (t, x, y, z) \to (t, x, -y, z)$ a Lorentz transformation? If so, why is it not considered with P and T as a discrete Lorentz transformation? If not, why not? Supplement: What about $Z : (t, x, y, z) \to (-t, -x, y, z)$?

Problem 2. (a) Show that for any $X_{\alpha\beta}$,

$$\frac{\partial}{\partial X_{\mu\nu}} X_{[\alpha\beta]} = \delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta]}. \tag{0.2}$$

(b) Show that,

$$\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\right) = F^{\mu\nu}\,, \tag{0.3}$$

where,

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{0.4}$$

Problem 3. (Schwartz 2.6) Lorentz invariance.

(a) Show that

$$\int_{-\infty}^{\infty} \mathrm{d}k^0\, \delta\left(k^2 - m^2\right)\, \theta(k^0) = \frac{1}{2\omega_k}, \tag{0.5}$$

where $\theta(x)$is the unit step function and $\omega_k := \sqrt{\mathbf{k}^2 + m^2}$.

(b) Show that the integration measure d^4k is Lorentz invariant.

(c) Finally, show that

$$\int \frac{\mathrm{d}^3k}{2\omega_k}, \tag{0.6}$$

is Lorentz invariant.

Dirac Delta Function, properties, Jackson - *Classical Electrodynamics*, 3rd ed

$$1^\circ(\text{definition}) \int \mathrm{d}x\, \delta(x)\, f(x) = f(0), \quad (\text{i.e. } \langle \delta, f\rangle = f(0))$$

$$2^\circ\ \delta(-x) = \delta(x), \quad 3^\circ\ \delta(ax) = \frac{\delta(x)}{|a|}, \quad 4^\circ \int \mathrm{d}x\, f(x)\delta(x-a) = f(a),$$

$$5^\circ\ \delta(g(x)) = \sum_{x_i \in g^{-1}(\{0\})} \frac{\delta(x - x_i)}{\left|\frac{\partial g(x)}{\partial x}\right|_{x=x_i}}, \quad x_i \in g^{-1}(\{0\}) = \{\, x \,|\, g(x) = 0\,\},$$

$$6^\circ\ \delta^3(\mathbf{x}) = \delta(x)\delta(y)\delta(z), \quad 7^\circ\ \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}xk}$$

Fourier transform in braket notation, Sakurai & Napolitano - *Modern Quantum Mechanics*

$$\langle \mathbf{x}'|\alpha\rangle = \psi_\alpha(\mathbf{x}'), \qquad \langle \mathbf{p}'|\alpha\rangle = \phi_\alpha(\mathbf{p}')$$

$$\int \mathrm{d}^3\mathbf{x}'\, |\mathbf{x}'\rangle\langle \mathbf{x}'| = \mathbb{1}, \quad \langle \mathbf{x}'|\mathbf{x}''\rangle = \delta^3(\mathbf{x}' - \mathbf{x}'')$$

$$\int \mathrm{d}^3\mathbf{p}'\, |\mathbf{p}'\rangle\langle \mathbf{p}'| = \mathbb{1}, \quad \langle \mathbf{p}'|\mathbf{p}''\rangle = \delta^3(\mathbf{p}' - \mathbf{p}'')$$

$$\langle \mathbf{x}'|\mathbf{p}'\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \cdot \exp\left(\mathrm{i}\mathbf{p}'\cdot\mathbf{x}'/\hbar\right)$$

$$\langle \mathbf{p}'|\mathbf{x}'\rangle = \langle \mathbf{x}'|\mathbf{p}'\rangle^* = \frac{1}{(2\pi\hbar)^{3/2}} \cdot \exp\left(-\mathrm{i}\mathbf{p}'\cdot\mathbf{x}'/\hbar\right)$$

$$\text{cf. discrete: } \sum_n |n\rangle\langle n| = \mathbb{1}, \quad \langle n|m\rangle = \delta_{nm}, \quad \langle n|\alpha\rangle = u_n$$

KGF Solution, discrete vs continuous version

$$\phi(x) = \sum\!\!\!\!\!\!\int_{k^2=\mu^2,\,\mathbf{k}\in\mathbb{R}^3} \tilde{\phi}(k)\mathrm{e}^{-\mathrm{i}kx}, \quad \sum\!\!\!\!\!\!\int_{\mathbf{k}} := \begin{cases} \lim\limits_{V\to\mathbb{R}^3} \frac{1}{V}\sum\limits_{\mathbf{k}}, & \text{for discrete } \mathbf{k}, \\ \int_{\mathbb{R}^3} \frac{\mathrm{d}^3k}{(2\pi)^3}, & \text{for continuous } \mathbf{k}. \end{cases}$$

with discrete $\mathbf{k}$ (see p. 4 and Eq. (1.48) on p. 12 in Mandl and Shaw) in a volume $V = L^3$:

$$\mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \quad n_1, n_2, n_3 \in \mathbb{Z}$$

so the number of states in the interval $(\mathbf{k}, \mathbf{k} + \mathrm{d}^3k)$ is,

$$\frac{V}{(2\pi)^3}\mathrm{d}^3k = \frac{V}{(2\pi)^3}|\mathbf{k}|^2\mathrm{d}k\mathrm{d}\Omega. \qquad \text{MS (1.47)}$$

The normalization volume V must drop out of all physically significant quantities such as transition rates etc.

$$\phi(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} \left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right].$$

In the continuous case and natural units (asymmetric version of the Fourier transform, as in Eq. (2.35) from Peskin and Schroeder),

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{(2\omega_{\mathbf{k}})^{1/2}} \left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right].$$