Solutions for Tutorials on Feynman amplitudes for QED

Olga Bessidskaia Bylund

February 24, 2017

1 Background

The order of expansion generally indicates how many vertices there are (with the exception of some forms of loop diagrams).

In QED, the expectation value of the first order expansion vanishes in vacumm due to energy-momentum conservation, e.g. going to the rest frame of initial electron, one sees that the total energy on the RHS would always be greater than for the LHS for the process $e^- \rightarrow e^- + \gamma$. One can obtain the other QED diagrams by rotating the external lines and for diagrams obtained in this way, this process (in isolation) will be forbidden too.

At second order expansion of QED, one can have virtual intermediate particles that do not obey the energy-momentum equation $p^2 = m^2$.

The integral $S_i^{(2)}$ for the transition from the initial state to the final state should annihilate the initial state particles and create the final state particles. For a process like an electron pair exchanging a photon, the momentum and spin will be diffent for the initial state and final state particles. The integral should contain an uncontracted part of the field for external particles (charged leptons or photons) and a contraction over two fields for the intermediate particle. The contracted fields connect the two vertices at x_1 and x_2.

The matrix element S_{fi} consists of the following components:

- The Feynman amplitude $\mathcal{M}$ - this contains the structure (u, v, ϵ_α, γ^μ).
- Delta functions that ensure energy-momentum conservation.
- Factors containing volume, mass and energy of the particles involved.

Drawing Feynman diagrams is a helpful tool for visualizing the process and calculating the matrix element. When drawing Feynman diagrams: if a fermion line enters a vertex (arrow in), a fermion line must also leave the vertex (arrow out). This reflects conservation of charge. Moreover, both lines must have the same flavour in QED (electron, muon, or tau).

2 7.1

For Bhabha scattering,

$$e^+(\mathbf{p}_1, r_1) + e^-(\mathbf{p}_2, r_2) \to e^+(\mathbf{p}_1', r_1) + e^-(\mathbf{p}_2', r_2), \tag{1}$$

we need to pick out the second order term in the S-matrix expansion that corresponds to 2 QED vertices with one contraction over the A_μ field. The initial electron positron pair should be annihilated and the final state electron positron pair should be created. The final state particles will have different momenta and spins from the intiial.

We start from the Feynman diagrams shown in Fig. 1. The process can come about either by the exchange of a photon or by annihilation and subsequent creation of lepton pairs.

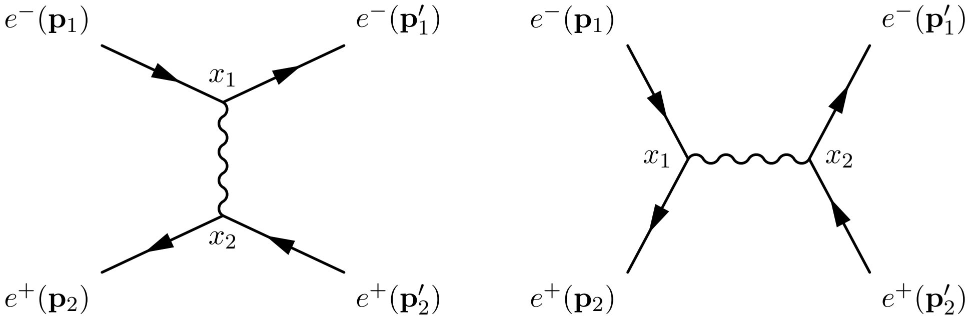

Figure 1: Bhabha scattering can occur either by exchange of a photon, shown to the left (S_a) or by annihilation and creation of the electron-positron pair, as shown to the right (S_b).

To identify which mathematical terms describe this process, we first look at Table 2, which is a reminder of how the expansion of the fermion field looks like.

Description	Operator	Wavefunction	Spinor
Annihilate e^-	c	ψ^+	u
Annihilate e^+	d	$\bar{\psi}^+$	$\bar{v}$
Create e^-	$c^\dagger$	$\bar{\psi}^-$	$\bar{u}$
Create e^+	$d^\dagger$	ψ^-	v

Table 1: Which part of the fields contain which operators and spinors and the enterpretation.

To calculate the matrix element, we need to contract over the photon fields. Let's look at the photon exchange diagram (a). At x_1, ψ^+ should annihilate the initial state electron and $\bar{\psi}^-$ should create the final state electron. At x_2, we need $\bar{\psi}^+$ to annihilate the initial state positron and ψ^- to create the final state positron.

Hence the relavant part for this process is:

$$S_a = -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 N[(\bar{\psi}^- \gamma^\alpha \psi^+)_{x_1} (\bar{\psi}^+ \gamma^\beta \psi^-)_{x_2}] iD_{F\alpha\beta}(x_1 - x_2), \quad (2)$$

$$S_b = -e^2 \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 N[(\bar{\psi}^- \gamma^\alpha \psi^-)_{x_1} (\bar{\psi}^+ \gamma^\beta \psi^+)_{x_2}] iD_{F\alpha\beta}(x_1 - x_2). \quad (3)$$

Here it has been taken into account, that one additional equation and diagram for S_a, S_b contributes, that differs by exchanging x_1 for x_2. As x_1 and x_2 are integrated over all of spacetime, x_1 and x_2 can be renamed in these diagrams. This gives a factor 2, which cancels the factor 1/2 from the sedond order S-matrix expansion.

First, looking at S_a only and putting it in normal order:

$$\begin{aligned} N[(\bar{\psi}^- \gamma^\alpha \psi^+)_{x_1} (\bar{\psi}^+ \gamma^\beta \psi^-)_{x_2}] = \\ = C \cdot N[(c^\dagger(\tilde{\mathbf{p}}_1') \bar{u}(\tilde{\mathbf{p}}_1') \mathrm{e}^{i\tilde{p}_1' x_1} \cdot \gamma^\alpha \cdot c(\tilde{\mathbf{p}}_1) u(\tilde{\mathbf{p}}_1) \mathrm{e}^{-i\tilde{p}_1 x_1}) \cdot \\ \cdot (d(\tilde{\mathbf{p}}_2) \bar{v}(\tilde{\mathbf{p}}_2) \mathrm{e}^{-i\tilde{p}_2 x_2} \cdot \gamma^\beta \cdot d^\dagger(\tilde{\mathbf{p}}_2') v(\tilde{\mathbf{p}}_2') \mathrm{e}^{i\tilde{p}_2' x_2})] = \\ C \exp[ix_1(\tilde{p}_1' - \tilde{p}_1)] \exp[ix_2(\tilde{p}_2' - \tilde{p}_2)] \cdot \\ \cdot \bar{u}(\tilde{\mathbf{p}}_1') \gamma^\alpha u(\tilde{\mathbf{p}}_1) \cdot \bar{v}(\tilde{\mathbf{p}}_2) \gamma^\beta v(\tilde{\mathbf{p}}_2') \cdot N[c^\dagger(\tilde{\mathbf{p}}_1') c(\tilde{\mathbf{p}}_1) d(\tilde{\mathbf{p}}_2) d^\dagger(\tilde{\mathbf{p}}_2')] \end{aligned} \quad (4)$$

The momenta $\tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2, \tilde{\mathbf{p}}_1', \tilde{\mathbf{p}}_2'$ are integrated over in the expansions of the electron fields while the momenta $\mathbf{p}_1, \mathbf{p}_2$ of the initial states and $\mathbf{p}_1', \mathbf{p}_2'$ of the final states are fixed. I am using the integral expansion of ψ, with $\sum_{\mathbf{p}}$ replaced by $\int \mathrm{d}^3\mathbf{p}$.

With

$$|i\rangle = c^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) |0\rangle, \quad (5)$$

$$|f\rangle = c^\dagger(\mathbf{p}_1') d^\dagger(\mathbf{p}_2') |0\rangle \to \langle f| = \langle 0| d(\mathbf{p}_2') c(\mathbf{p}_1'), \quad (6)$$

the part of $\langle f| S |i\rangle$ containing the operators is:

$$\begin{aligned} \langle 0| d(\mathbf{p}_2') c(\mathbf{p}_1') \cdot N[c^\dagger(\tilde{\mathbf{p}}_1') c(\tilde{\mathbf{p}}_1) d(\tilde{\mathbf{p}}_2) d^\dagger(\tilde{\mathbf{p}}_2')] \cdot c^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) |0\rangle = \\ = -\langle 0| d(\mathbf{p}_2') c(\mathbf{p}_1') \cdot [c^\dagger(\tilde{\mathbf{p}}_1') d^\dagger(\tilde{\mathbf{p}}_2') d(\tilde{\mathbf{p}}_2) c(\tilde{\mathbf{p}}_1)] c^\dagger(\mathbf{p}_1) d^\dagger(\mathbf{p}_2) |0\rangle = \\ = -\langle 0| d(\mathbf{p}_2') \delta_{\mathbf{p}_1', \tilde{\mathbf{p}}_1'} \cdot d^\dagger(\tilde{\mathbf{p}}_2') d(\tilde{\mathbf{p}}_2) \delta_{\mathbf{p}_1, \tilde{\mathbf{p}}_1} \cdot d^\dagger(\mathbf{p}_2) |0\rangle = \\ = -\delta_{\mathbf{p}_2', \tilde{\mathbf{p}}_2'} \delta_{\mathbf{p}_1', \tilde{\mathbf{p}}_1'} \delta_{\mathbf{p}_1, \tilde{\mathbf{p}}_1} \delta_{\mathbf{p}_2, \tilde{\mathbf{p}}_2} \end{aligned} \quad (7)$$

Above, the anticommutation relations of the ladder operators are used together with that an annihilation operator acting on $|0\rangle$ gives zero. Note that it is important that to get the order of the operators in $|f\rangle$ and $|i\rangle$ consistent between all contributions to $\mathcal{M}$ to get the correct sign between the different terms.

Putting everything together fixes the momenta to those of the initial or final state particles:

$$\langle f|\, S_a \,|i\rangle = +\int \mathrm{d}^4x_1\mathrm{d}^4x_2\, e^2 \prod_j \left(\frac{m}{VE_{\mathbf{p}_j}}\right)^{1/2} \cdot \frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, iD_F^{\alpha\beta}(k)\mathrm{e}^{-ik(x_1-x_2)}$$
$$\exp[ix_1(p_1'-p_1)]\exp[ix_2(p_2'-p_2)]\cdot \bar{u}(\mathbf{p}_1)\gamma^\alpha u(\mathbf{p}_1')\bar{v}(\mathbf{p}_1)\gamma^\beta v(\mathbf{p}_2). \quad (8)$$

Here the product is over all four uncontracted fermion fields.

The exponentials give δ-funcitons that fix the four-momenta thus ensuring energy-momentum conservation:

$$\int \mathrm{d}^4x_1\ \exp[ix_1(p_1'-p_1-k)]\int \mathrm{d}^4x_2\ \exp[ix_2(p_2'-p_2+k)] =$$
$$= (2\pi)^4(2\pi)^4\delta(p_2'-p_2+p_1'-p_1) \quad (9)$$

One of the $(2\pi)^4$ cancels that in the photon propagator. Then:

$$\langle f|\, S_a \,|i\rangle = \frac{m_e^2e^2}{\sqrt{E_1E_2E_1'E_2'}}\cdot iD_{F\alpha\beta}$$
$$\delta(p_2'-p_2+p_1'-p_1)\bar{u}(\mathbf{p}_1)\gamma^\alpha u(\mathbf{p}_1')\bar{v}(\mathbf{p}_2')\gamma^\beta v(\mathbf{p}_2) \quad (10)$$

The integral over k disappeared when implementing the second δ function. The Feynman amplitude for S_a is

$$\mathcal{M}_a = e^2 iD_{F\alpha\beta}(k=p_1'-p_1)\bar{u}(\mathbf{p}_1)\gamma^\alpha u(\mathbf{p}_1')\bar{v}(\mathbf{p}_2')\gamma^\beta v(\mathbf{p}_2). \quad (11)$$

Using eq (7.24 b) and taking $\epsilon \to 0$: $D_{F\alpha\beta} = -\frac{\eta_{\alpha\beta}}{k^2}$.:

$$\mathcal{M}_a = -\frac{ie^2}{(p_1'-p_1)^2}\bar{u}(\mathbf{p}_1)\gamma^\alpha u(\mathbf{p}_1')\bar{v}(\mathbf{p}_2')\gamma_\alpha v(\mathbf{p}_2). \quad (12)$$

Doing the corresponding calculation forS_b would give:

$$\mathcal{M}_b = +\frac{ie^2}{(p_1-p_1')^2}\bar{u}(\mathbf{p}_1)\gamma^\alpha v(\mathbf{p}_2)\bar{v}(\mathbf{p}_2')\gamma^\beta u(\mathbf{p}_1'). \quad (13)$$

3 Problem 7.3

The Lagrangian given for the spin zero boson is:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I, \quad (14)$$
$$\mathcal{L}_I = g[\phi(x)]^4/4! \quad (15)$$

This is an example of ϕ^4 theory. We are to write down the S-matrix expansion for this theory and select the term that corresponds to

$$B(\mathbf{k}_1) + B(\mathbf{k}_2) \to B(\mathbf{k}_1') + B(\mathbf{k}_2') \quad (16)$$

Solution:
We are interested in the process that annihilates two particles and creates two particles. In ϕ^4 theory, we can have four external fields at each vertex. Therefore, we can stay at first order term in the expansion of the S-matrix without violating energy-momentum conservation. For the first order:

$$S^{(1)} = -i \int \mathrm{d}^4 x \, g \, N[\phi(x)]^4/4! =$$
$$-\frac{ig}{4!} \int \mathrm{d}^4 x N[(\phi^+ + \phi^-)_x (\phi^+ + \phi^-)_x (\phi^+ + \phi^-)_x (\phi^+ + \phi^-)_x] \qquad (17)$$

There are 4! different ways to assign each of the fields to one of the terms in Eq (16). (Alternatively, one can think of it as there being 4 choose 2 ways to assign the fields to either initial state or final state and then 2 choices each for how to assign them to which initial and final state momentum.) This cancels the 1/4! prefactor, leaving

$$\tilde{S}^{(1)} = -ig \int \mathrm{d}^4 x \, (\phi^- \phi^- \phi^+ \phi^+)_x \qquad (18)$$

Since we are dealing with scalars, we need not keep track of the number of adjacent permutations involved while writing the product in normal order. I introduced the notation $\tilde{S}^{(1)}$ for the terms that are relevant for the process considered. We know that the other terms of $S^{(1)}$ will vanish in $\langle f | \, S^{(1)} \, | i \rangle$. We evaluate $\langle f | \, S^{(1)} \, | i \rangle$, using the expansion given in Eq. (3.7) in Mandl and Shaw and setting $c = \hbar = 1$.

$$\langle f | \, S^{(1)} \, | i \rangle = -ig \int \mathrm{d}^4 x \prod_j \frac{1}{\sqrt{2V\omega_j}} \mathrm{e}^{ik_1' x} \mathrm{e}^{ik_2' x} \mathrm{e}^{-ik_1 x} \mathrm{e}^{-ik_2 x} \qquad (19)$$

where each of the four external field contributes one factor to the product written with $\prod$. Evaluating the x-dependent part, calling it $f(x)$,

$$f(x) = \int \mathrm{d}^4 x \, \exp[ix(k_1' + k_2' - k_1 - k_2)] = (2\pi)^4 \delta(k_1' + k_2' - k_1 - k_2) \qquad (20)$$

The Feynman amplitude is identified as

$$\mathcal{M} = -ig \qquad (21)$$

and the expression can be written as

$$\langle f | \, S^{(1)} \, | i \rangle = (2\pi)^4 \delta(k_1' + k_2' - k_1 - k_2) \prod_j \frac{1}{\sqrt{2V\omega_j}} \mathcal{M} \qquad (22)$$

Note that this lagrangian makes possible first order processes in vacuum. If an external field is applied, one initial ϕ can decay into three ϕ particles.

4 Problem 7.5

The Lagrangian is given by

$$\mathcal{L}(x) = \mathcal{L}_0(x) \quad + \quad \mu U(\mathbf{x})\phi^2(x) =$$
$$\frac{1}{2}(\phi_{,\alpha}(x)\phi^{,\alpha}(x) - \mu^2\phi^2(x)) \quad + \quad \mu U(\mathbf{x})\phi^2(x) \tag{23}$$

First derive the equations of motion. The dependence on the four-vector x will not be written out explicitly below. The expressions $\phi_{,\alpha}$ and $\partial_\alpha\phi$ are equivalent.

Solution:

Use the Euler-Lagrange equation, Eq. (2.16) in Mandl and Shaw:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right] = 0. \tag{24}$$

The first term is

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\mu^2\phi + 2\mu U(\mathbf{x})\phi. \tag{25}$$

The second term is

$$\partial_\nu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\right] = \partial_\nu\partial^\nu\phi = \Box\phi \tag{26}$$

Putting the expressions for the terms into the Euler-Lagrange equation, we get

$$(\Box + \mu^2)\phi = 2\mu U(\mathbf{x})\phi \tag{27}$$

Next, we should calculate the S-matrix element to lowest order for an incoming boson with momentum k_i to be scattered to a state with mometum k_f. We can have a non-zero contribution from the first order even though this is ϕ^2 theory - momentum can be conserved at the vertex through interaction with the potential.

The first order term in the S-matrix expansion is

$$S^{(1)} = i\int \mathrm{d}^4x\, N(\mu U(\mathbf{x})\phi^2(x)) \tag{28}$$

There is no minus sign since $\mathcal{H}$ appears in the expansion and $\mathcal{H} = -\mathcal{L}_I$. This minus sign cancels the one from Eq. (7.1) in Mandl and Shaw. Picking out only the term that has one annihilation and one creation operator and noting that this can come from either of the two fields ϕ,

$$\tilde{S}^{(1)} = 2i\int \mathrm{d}^4x\, (\mu U(\mathbf{x})\phi^-(x)\phi^+(x)) \tag{29}$$

I rename the four-momenta $k = k_i$, $k' = k_f$. Inserting the expansion of the ϕ field:

$$\langle k'| \, S^{(1)} \, |k\rangle = 2i\mu \int \mathrm{d}^4x \, U(\mathbf{x}) \prod_j (\frac{1}{2V\omega_{k_j}})^{1/2} \mathrm{e}^{ik'x} \mathrm{e}^{-ikx} \tag{30}$$

The x-dependent part needs to be split it into a space-part and a time-part.

$$f(x) = \int \mathrm{d}^4x \, U(\mathbf{x}) \exp[ix(k' - k)] = (2\pi)\delta(\omega' - \omega) \int \mathrm{d}^3 \, \mathbf{x} U(\mathbf{x}) \mathrm{e}^{-i\mathbf{q}\mathbf{x}}, \quad \mathbf{q} = \mathbf{k} - \mathbf{k}' \tag{31}$$

The expression in the integral is the Fourier transform of $U(\mathbf{x})$, denoted $\tilde{U}(\mathbf{q})$. The S-matrix element is

$$\langle k'| \, S^{(1)} \, |k\rangle = \frac{i2\pi\delta(\omega' - \omega)}{(2V\omega)^{1/2}(2V\omega')^{1/2}} 2\mu\tilde{U}(\mathbf{k}' - k). \tag{32}$$