Tutorial on Cross Sections, Solutions

Please do not simply quote the results derived here for the problem sets. We want you to go through the steps yourselves.

1 Problem: Cross-Section in Lab Frame

As the second particle is approximately stationary, we have $\mathbf{p}_2 = 0$ so that $p_2^2 = E_2^2 = m_2^2$.

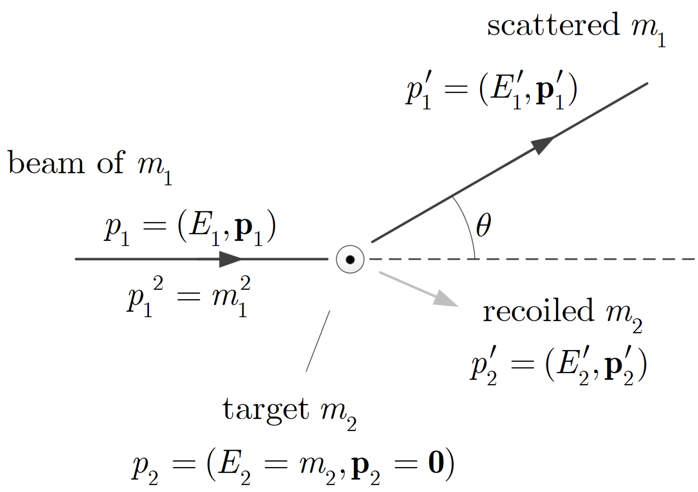

Figure 1. Kinematics of $(m_1 m_2)$ scattering in the laboratory frame

Let further the spherical polar coordinate axis $\hat{\mathbf{z}}$ be in direction of $\mathbf{p}_1$, so that θ' is the scattering angle where (θ', ϕ') are the polar angles of $\mathbf{p}_1'$ Then the polarized differential cross-section for this process is given by (MS-Eqs. (8.15) and (8.12b))

$$\mathrm{d}\sigma = f(p_1', p_2')\, |\mathbf{p}_1'|^2\, \mathrm{d}\Omega_1' \left[\frac{\partial (E_1' + E_2')}{\partial\, |\mathbf{p}_1'|}\right]^{-1}, \tag{1.1}$$

$$f(p_1', p_2') \equiv \frac{1}{64\pi^2 v_{\mathrm{rel}} E_1 E_2 E_1' E_2'} \left(\prod_l (2m_l)\right) |\mathcal{M}|^2, \tag{1.2}$$

where $\mathcal{M}$ is the Feynman amplitude for this transition (to be kept general in this problem), $\mathrm{d}\Omega_1' = \sin\theta' \mathrm{d}\theta' \mathrm{d}\phi'$ is the corresponding element of solid angle, and the partial derivative is evaluated with the polar angles (θ', ϕ') of the vector $\mathbf{p}_1'$ constant. The relative velocity between the incident particle and the target in units of c in the Lab frame (MS-Eq. (8.10b)) is:

$$v_{\mathrm{rel}} = \frac{|\mathbf{p}_1|}{E_1}\,.$$

Substituting v_{rel} and $f(p_1', p_2')$ into $\mathrm{d}\sigma$, also noting $E_2 = m_2$, we get:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_1'} = \frac{1}{64\pi^2 \frac{|\mathbf{p}_1|}{E_1} E_1 E_2 E_1' E_2'} 16 m_1^2 m_2^2\, |\mathbf{p}_1'|^2 \left[\frac{\partial (E_1' + E_2')}{\partial\, |\mathbf{p}_1'|}\right]^{-1} |\mathcal{M}|^2, \tag{1.3}$$

$$= \frac{1}{4\pi^2} \frac{|\mathbf{p}_1'|^2}{|\mathbf{p}_1|} \frac{m_1^2 m_2}{E_1' E_2'} \left[\frac{\partial (E_1' + E_2')}{\partial\, |\mathbf{p}_1'|}\right]^{-1} |\mathcal{M}|^2\,. \tag{1.4}$$

What remains is to evaluate the partial derivative. Use the energy-momentum relation:

$$E_1'^2 = m_1^2 + |\mathbf{p}_1'|^2 \,, \tag{1.5}$$
$$E_2'^2 = m_2^2 + |\mathbf{p}_2'|^2 \,. \tag{1.6}$$

From the conservation of momentum:

$$\mathbf{p}_2' = \mathbf{p}_1 - \mathbf{p}_1' \,, \tag{1.7}$$
$$|\mathbf{p}_2'|^2 = |\mathbf{p}_1|^2 + |\mathbf{p}_1'|^2 - 2\,|\mathbf{p}_1|\,|\mathbf{p}_1'|\cos\theta' \,. \tag{1.8}$$

This gives

$$E_1' = \left(m_1^2 + |\mathbf{p}_1'|\right)^{1/2} , \tag{1.9}$$
$$E_2' = \left(m_2^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_1'|^2 - 2\,|\mathbf{p}_1|\,|\mathbf{p}_1'|\cos\theta'\right)^{1/2} \,. \tag{1.10}$$

Taking the partial derivative with respect to $|\mathbf{p}_1'|$ while keeping θ', ϕ' fixed, we get:

$$\left(\frac{\partial E_1'}{\partial\,|\mathbf{p}_1'|}\right)_{\theta'\phi'} = \frac{1}{2}\left(m_1^2 + |\mathbf{p}_1'^2)\right)^{-1/2} \cdot 2\,|\mathbf{p}_1'| = \frac{|\mathbf{p}_1'|}{E_1'} \,, \tag{1.11}$$

$$\left(\frac{\partial E_2'}{\partial\,|\mathbf{p}_1'|}\right)_{\theta'\phi'} = \frac{1}{2}\left(m_2^2 + |\mathbf{p}_1|^2 + |\mathbf{p}_1'|^2 - 2\,|\mathbf{p}_1|\,|\mathbf{p}_1'|\cos\theta'\right)^{-1/2} \cdot$$
$$\cdot 2\left(|\mathbf{p}_1'| - |\mathbf{p}_1|\cos\theta'\right) = \frac{|\mathbf{p}_1'| - |\mathbf{p}_1|\cos\theta'}{E_2'} \,, \tag{1.12}$$

which yields (using conservation of energy in the next to last step)

$$\left(\frac{\partial(E_1' + E_2')}{\partial\,|\mathbf{p}_1'|}\right)_{\theta'\phi'} = \frac{|\mathbf{p}_1'|}{E_1'} + \frac{|\mathbf{p}_1'| - |\mathbf{p}_1|\cos\theta'}{E_2'} \tag{1.13}$$
$$= \frac{(E_1' + E_2')\,|\mathbf{p}_1'| - E_1'\,|\mathbf{p}_1|\cos\theta'}{E_1'E_2'} \tag{1.14}$$
$$= \frac{(E_1 + E_2)\,|\mathbf{p}_1'| - E_1'\,|\mathbf{p}_1|\cos\theta'}{E_1'E_2'} \tag{1.15}$$
$$= \frac{(E_1 + m_2)\,|\mathbf{p}_1'| - E_1'\,|\mathbf{p}_1|\cos\theta'}{E_1'E_2'} \,. \tag{1.16}$$

Then

$$\frac{1}{E_1'E_2'}\left[\left(\frac{\partial(E_1' + E_2')}{\partial\,|\mathbf{p}_1'|}\right)_{\theta'\phi'}\right]^{-1} = \left[(E_1 + m_2)\,|\mathbf{p}_1'| - E_1'\,|\mathbf{p}_1|\cos\theta'\right]^{-1} . \tag{1.17}$$

After substituting the last expression back into Eq. (1.4), we get the final expression for the differential cross-section for the scattering of two particles of masses m_1, m_2 in the laboratory frame:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'}\right)_{\mathrm{Lab}} = \frac{m_1^2 m_2 |\mathbf{p}_1'|^2}{4\pi^2 \,|\mathbf{p}_1| \cdot ((E_1+m_2)\,|\mathbf{p}_1'| - E_1'\,|\mathbf{p}_1|\cos\theta')} \,|\mathcal{M}|^2 \,. \tag{1.18}$$

2 Problem: e^-e^+ production in EM field

Initial state: $|i\rangle = |0\rangle$. Final state: $|f\rangle = c^\dagger(\mathbf{p}_1, r)d^\dagger(\mathbf{p}_2, s)\,|0\rangle$ with $\mathbf{p}_1$ being the momentum of the electron and $\mathbf{p}_2$ of the positron. The potential is $A_\mu = (0, 0, ae^{-i\omega t}, 0)$. Then

$$\langle f|\, S^{(1)}\,|i\rangle = ie\,\langle f| \int \mathrm{d}^4x N[\bar{\psi}\not{A}_\mu\psi]\,|0\rangle \tag{2.1}$$

From anticommutation relations of the annihilation and creation operators, the only term that does not vanish and that corresponds to the pair production is the following:

$$\langle f|\, S_{fi}\,|0\rangle = \frac{ie}{V}\int \mathrm{d}^4x \sum_{\mathbf{p}_1,r\mathbf{p}_2 s} \sqrt{\frac{m}{E_{q_1}}}\sqrt{\frac{m}{E_{q_2}}}\,\cdot$$
$$\cdot\,\langle 0|\, d(\mathbf{p}_2, s)c(\mathbf{p}_1, r)\cdot c^\dagger(\tilde{\mathbf{p}}_1, \tilde{r})\mathrm{e}^{i\tilde{p}_1 x}\bar{u}(\tilde{\mathbf{p}}_1, \tilde{r})\cdot\gamma^2 a\mathrm{e}^{-i\omega t}\cdot d^\dagger(\tilde{\mathbf{p}}_2, \tilde{s})\mathrm{e}^{i\tilde{p}_2 x}v(\tilde{\mathbf{p}}_2, \tilde{s})\,|0\rangle \tag{2.2}$$

Following the same procedure as in the tutorial on Feynman amplitudes, this reduces to:

$$\langle f|\, S^{(1)}\,|i\rangle = \frac{iea}{V}\sqrt{\frac{m}{E_1}}\sqrt{\frac{m}{E_2}}\sum\int \mathrm{d}^4x\bar{u}(\mathbf{p}_1, r)\gamma^2 v(\mathbf{p}_2, s)\mathrm{e}^{ip_1x+ip_2x-i\omega t} =$$
$$= \frac{ieam}{V}\frac{1}{\sqrt{E_1E_2}}(2\pi)^4\delta^3(\mathbf{p}_1+\mathbf{p}_2)\delta(E_2+E_1-\omega)\bar{u}(\mathbf{p}_1, r)\gamma^2 v(\mathbf{p}_2, s)\,. \tag{2.3}$$

So, the electron and positron have momenta that are equal in magnitude and opposite in direction. The overall energy (including that of the photon) is conserved. The Feynman amplitude is identified as:

$$\mathcal{M} = iea\cdot\bar{u}(\mathbf{p}_1, r)\gamma^2 v(\mathbf{p}_2, s)\,. \tag{2.4}$$

A note on signs: There would be a reverse sign on $\mathcal{M}$ if the final state was defined as $|f\rangle = d^\dagger(\mathbf{p}_2, s)c^\dagger(\mathbf{p}_1, r)\,|0\rangle$ instead. What matters for the computation of physical quantities, such as cross section, which is proportional to $|\mathcal{M}|^2$, is the relative sign on the different contributions $\mathcal{M}_i$ for the same process. So the initial and final states must be defined with the ladder operators writen in the same order for all contributing $\mathcal{M}_i$.

Summing over the polarization states in the final state gives the expression below. Note that in $\mathcal{M}^*$, $\gamma^{2\dagger} = \gamma^0\gamma^2\gamma^0$ and $\bar{u}^\dagger = \gamma^0 u$. See the next problem for more detail (with B in place of γ^2). Together this gives:

$$|\mathcal{M}|^2 = e^2a^2\sum_{rs}\bar{u}(\mathbf{p}_1, r)\gamma^2 v(\mathbf{p}_2, s)\cdot\bar{v}(\mathbf{p}_2, s)\gamma^2 u(\mathbf{p}_1, r) =$$

$$= e^2 a^2 \mathrm{Tr}[\frac{\not p_2' - m}{2m}\gamma^2 \frac{\not p_1 + m}{2m}\gamma^2] =$$
$$= \frac{e^2 a^2}{4m^2}[\mathrm{Tr}(\not p_2 \gamma^2 \not p_1 \gamma^2) - m^2 \mathrm{Tr}(\gamma^2\gamma^2)]. \tag{2.5}$$

In the last line, the terms with an odd number of γ matrices have been omitted, since the trace of such products vanishes. Below, the primes on p_1, p_2 are dropped for brevity. Using trace relations for γ matrices from the appendix in Mandl and Shaw (A.17) ,

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta}, \tag{2.6}$$
$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}) \tag{2.7}$$
$$\mathrm{Tr}(\mathrm{odd}\,\#\,\gamma\mathrm{s}) = 0 \tag{2.8}$$

and multiplying p_μ out of the trace, we get:

$$\mathrm{Tr}(\gamma^2\gamma^2) = 4g^{22} = -4, \tag{2.9}$$
$$\mathrm{Tr}(\not p_2 \gamma^2 \not p_1 \gamma^2) = 4((p_2)_\mu (p_1)_\nu)(g^{\mu 2}g^{\nu 2} - g^{\mu\nu}g^{22} + g^{\mu 2}g^{2\nu}), =$$
$$= 4[2(p_2)^2(p_1)^2 + p_{2\mu}p_1^\mu)] \tag{2.10}$$

To go to the differential cross section $\mathrm{d}\sigma$, we use a modified version of (8.3) and (8.8) - here there is no division by V/v_{rel}. The reason for this is that we have the decay of a particle and hence no flux of any colliding particles.

$$\mathrm{d}\sigma = w\frac{V\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3}\frac{V\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3}, \quad w = |S_{fi}|^2/T \tag{2.11}$$

We have

$$w = \frac{1}{T}[\frac{1}{V}\frac{2m}{\sqrt{E_1E_2}}(2\pi)^4\delta^3(\mathbf{p}_2 - \mathbf{p}_1)\delta(E_2 + E_1 - \omega)]^2|\mathcal{M}|^2 =$$
$$= \frac{1}{T}\frac{4m^2}{V^2E_1E_2}(2\pi)^4(TV)\delta^3(\mathbf{p}_2 - \mathbf{p}_1)\delta(E_2 + E_1 - \omega)|\mathcal{M}|^2 \tag{2.12}$$

Inserting w and $\mathrm{d}\sigma$, we get:

$$\mathrm{d}\sigma = \int\int \mathrm{d}^3\mathbf{p}_1\mathrm{d}^3\mathbf{p}_2\frac{1}{(2\pi)^2}\delta^3(\mathbf{p}_2 - \mathbf{p}_1)\delta(E_2 + E_1 - \omega)\cdot$$
$$\cdot\frac{e^2a^2}{2\pi^2E_1^2}4[2(p_2)^2(p_1)_2 + (p_2p_1) + m^2] \tag{2.13}$$

Writing the momenta in spherical coordinates in the CoM frame gives:

$$p_1^\mu = (E_1,\, |\mathbf{p}_1|\sin\theta\cos\phi,\, |\mathbf{p}_1|\sin\theta\sin\phi,\, |\mathbf{p}_1|\cos\theta) \tag{2.14}$$
$$p_2^\mu = (E_1,\, -|\mathbf{p}_1|\sin\theta\cos\phi,\, -|\mathbf{p}_1|\sin\theta\sin\phi,\, -|\mathbf{p}_1|\cos\theta). \tag{2.15}$$

Then

$$(p_1)^\mu(p_2)_\mu = E_1^2 + |\mathbf{p}|^2\,, \tag{2.16}$$
$$(p_1)^2(p_2)^2 = -|\mathbf{p}_1|^2 \sin^2\theta \sin^2\phi\,. \tag{2.17}$$

Then eq. (2.5) becomes:

$$\frac{e^2a^2}{2\pi^2 E_1^2}(E_1^2 + |\mathbf{p}_1|^2 + m^2 - 2|\mathbf{p}_1|^2 \sin^2\theta\sin^2\phi) \tag{2.18}$$

So the expression for the differential cross section becomes:

$$\mathrm{d}\sigma = \int \mathrm{d}^3\mathbf{p_1}\frac{e^2a^2}{2E_1^2(2\pi)^2}\delta(2E_1-\omega)(E_1^2 + |\mathbf{p}_1|^2 + m^2 - 2|\mathbf{p}_1|^2\sin^2\theta\sin^2\phi) =$$
$$\int \mathrm{d}^3\mathbf{p_1}\frac{e^2a^2}{2VE_1^2(2\pi)^2}\delta(E_1-\omega/2)[2E_1^2 - 2(E_1^2-m^2)\sin^2\theta\sin^2\phi] \tag{2.19}$$

Now we integrate over $\mathrm{d}^3\mathbf{p}_1$ and use that $\mathrm{d}^3\mathbf{p}_1 = |\mathbf{p}_1|^2\mathrm{d}|\mathbf{p}_1|\mathrm{d}\Omega = 2|\mathbf{p}_1|E_1\mathrm{d}E_1\mathrm{d}\Omega$:

$$\sigma = \frac{e^2a^2}{(2\pi)^2}\int\int \mathrm{d}E_1\mathrm{d}\Omega\frac{|\mathbf{p}_1|E_1}{2E_1^2}\delta(E_1-\omega/2)[2E_1^2 - 2(E_1^2-m^2)\sin^2\theta\sin^2\phi] \tag{2.20}$$

With $|\mathbf{p}_1| = \sqrt{E_1^2-m^2}$ and $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$:

$$\sigma = \frac{e^2a^2}{(2\pi)^2}\int_0^\pi \sin\theta\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi\frac{\sqrt{\omega^2/4-m^2}}{\omega/2}\{(\omega/2)^2 - [(\omega/2)^2-m^2]\sin^2\theta\sin^2\phi\} =$$
$$= \frac{e^2a^2}{2\pi^2}\frac{\sqrt{\omega^2/4-m^2}}{\omega}\{\frac{\omega^2}{4}4\pi - [\frac{\omega^2}{4}-m^2]4\pi/3\} \tag{2.21}$$

In the last step I used that $\int_0^\pi sin^3\theta\mathrm{d}\theta = \frac{4}{3}$, $\int_0^{2\pi} sin^2\phi\mathrm{d}\phi = \pi$, as found by Wolfram Alpha. This simplifies to:

$$\sigma = \frac{e^2a^2}{2\pi^2}\frac{\sqrt{\omega^2/4-m^2}}{\omega}[\omega^2\pi(1-1/3) + m^2 4\pi/3] = \frac{e^2a^2}{3\pi\omega}\sqrt{\omega^2/4-m^2}(\omega^2+2m^2) \tag{2.22}$$

Note that for this process to happen, we must have

$$\omega^2/4 > m^2 \to \hbar\omega > 2m \to E > 2m. \tag{2.23}$$

Also, this first order QED process can have because of interactions with an external field. There is no momentum conservation in this formulation of the problem, but really the momentum is taken from the field.

3 Spin-sums Lemma *(Casimir's trick for spin-sums)*

Proof. We will use the energy projection operators MS-Eq. (A.31)

$$\Lambda^{\pm}(\mathbf{p}) = \frac{\pm\not{p} + m}{2m}\,, \tag{3.1}$$

which are given in the component form as MS-Eqs. (A.34) alt. (8.24)

$$\Lambda^{+}_{\alpha\beta}(\mathbf{p}) = \sum_r u_{r\alpha}(\mathbf{p})\bar{u}_{r\beta}(\mathbf{p}) = \left(\frac{\not{p} + m}{2m}\right)_{\alpha\beta}\,, \tag{3.2}$$

$$-\Lambda^{-}_{\alpha\beta}(\mathbf{p}) = \sum_r v_{r\alpha}(\mathbf{p})\bar{v}_{r\beta}(\mathbf{p}) = \left(\frac{\not{p} - m}{2m}\right)_{\alpha\beta}\,. \tag{3.3}$$

We will only prove the first of the four identies (listed in the document on Problems) as proving the other relations follows a procedure with the same pattern. Expanding the left hand side, we obtain

$$X = \sum_{r,r'} \left(\bar{u}_{r'}(\mathbf{p}')\,A\,v_r(\mathbf{p})\right)\left(\bar{u}_{r'}(\mathbf{p}')\,B\,v_r(\mathbf{p})\right)^{\dagger} \tag{3.4}$$

$$= \sum_{r,r'} \left(u^{\dagger}_{r'}(\mathbf{p}')\gamma^0\,A\,v_r(\mathbf{p})\right)\left(u^{\dagger}_{r'}(\mathbf{p}')\gamma^0\,B\,v_r(\mathbf{p})\right)^{\dagger} \tag{3.5}$$

$$= \sum_{r,r'} \left(u^{\dagger}_{r'}(\mathbf{p}')\gamma^0\,A\,v_r(\mathbf{p})\right)\left(v^{\dagger}_{r}(\mathbf{p})\,B^{\dagger}\,\gamma^{0\dagger}u^{\dagger\dagger}_{r'}(\mathbf{p}')\right) \tag{3.6}$$

$$/\ \text{between } u^{\dagger}\text{and } B^{\dagger}\text{insert } \gamma^0\gamma^0 = 1\ / \tag{3.7}$$

$$= \sum_{r,r'} \left(u^{\dagger}_{r'}(\mathbf{p}')\gamma^0\,A\,v_r(\mathbf{p})\right)\left(v^{\dagger}_{r}(\mathbf{p})\,(\gamma^0)^2\,B^{\dagger}\,\gamma^{0}u_{r'}(\mathbf{p}')\right) \tag{3.8}$$

$$\sum_{r,r'} \left(\bar{u}_{r'}(\mathbf{p}')\,A\,v_r(\mathbf{p})\right)\left(\bar{v}_{r}(\mathbf{p})\,\gamma^0 B^{\dagger}\gamma^{0}\,u_{r'}(\mathbf{p}')\right) \tag{3.9}$$

$$/\ \text{write out the spinor indices explicitly}\ /$$

$$= \sum_{r,r'} \left(\bar{u}_{r'\mu_1}(\mathbf{p}')A_{\mu_1\mu_2}v_{r\mu_2}(\mathbf{p})\right)\left(\bar{v}_{r\mu_3}(\mathbf{p})\,\gamma^0_{\mu_3\mu_4}B^{\dagger}_{\mu_4\mu_5}\gamma^0_{\mu_5\mu_6}u_{r'\mu_6}(\mathbf{p}')\right) \tag{3.10}$$

$$/\ \text{we can freely move the spinor components}\ /$$

$$= \left(\sum_{r'} u_{r'\mu_6}(\mathbf{p}')\bar{u}_{r'\mu_1}(\mathbf{p}')\right)A_{\mu_1\mu_2}\left(\sum_{r} v_{r\mu_2}(\mathbf{p})\bar{v}_{r\mu_3}(\mathbf{p})\right)\gamma^0_{\mu_3\mu_4}B^{\dagger}_{\mu_4\mu_5}\gamma^0_{\mu_5\mu_6} \tag{3.11}$$

$$/\ \text{use the energy projection operators}\ /$$

$$= \left(\frac{\not{p}' + m}{2m}\right)_{\mu_6\mu_1}A_{\mu_1\mu_2}\left(\frac{\not{p}' - m}{2m}\right)_{\mu_2\mu_3}\gamma^0_{\mu_3\mu_4}B^{\dagger}_{\mu_4\mu_5}\gamma^0_{\mu_5\mu_6} \tag{3.12}$$

$$= \mathrm{Tr}\left[\frac{\not{p}' + m}{2m}\,A\,\frac{\not{p}' - m}{2m}\,\gamma^0 B^{\dagger}\gamma^0\right] \tag{3.13}$$

$$= \frac{1}{4m^2}\mathrm{Tr}\left[\,(\not{p}' + m)\,A\,(\not{p} - m)\,\gamma^0 B^{\dagger}\gamma^0\,\right] \tag{3.14}$$

$$= \frac{1}{4m^2}\mathrm{Tr}\left[\,(\not{p}' + m)\,A\,(\not{p} - m)\,\tilde{B}\,\right]\quad\square \tag{3.15}$$

4 Extra - interpretation of cross section in collider experiments

The cross section is a measure of the probability for a certain scattering process in particle physics. It is given in units of barn for historical reasons, with $1b = 10^{-2}\,\mathrm{m}^2$. The prefix pico- is usually attached to describe processes in particle physics (1 pb = 10^{-12} b).

As a naive picture, the area in m^2 corresponds to the magnitude of the probability that two particles approaching each other, each with an uncertainty of 1m in x- and y-direction will interact in a certain way.

In practice, a beam of many particles is brought to collide with another beam. The cross section contains the information that is only dependent on the colliison energy. The information about how many particles are present in a beam N_1, N_2, how often two beams collide (f) and the dimensions of the beam (σ_x, σ_y) is needed to know how often a collision will occur. This information, which is dependent on the accelerator employed, is stored in the luminosity $\mathcal{L}$:

$$\mathcal{L} = \frac{N_a N_b f}{4\pi\sigma_x\sigma_y}. \tag{4.1}$$

The rate for the process in units of per second is

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \sigma L. \tag{4.2}$$

At the LHC, $\sigma_x = \sigma_y \sim 60\mu m$. With one particle in each bunch brought to collide each second, the rate would be

$$\frac{\mathrm{d}R}{\mathrm{d}t} \sim 10^{-33}/s \tag{4.3}$$

and one would need to wait for much longer than the age of the Universe to observe the interaction. With 25 ns between each bunch crossing, $f = 40 \cdot 10^6$ /s and

$$\frac{\mathrm{d}R}{\mathrm{d}t} \sim 10^{-27}/s, \tag{4.4}$$

nearer the age of the universe. And by putting 10^{11} protons in each bunch at the LHC a rate of

$$\frac{\mathrm{d}R}{\mathrm{d}t} \sim 10^{-3}/s \tag{4.5}$$

is obtained, with a few ineractions per hour for a process with $\sigma = 1$ pb.

The tighter one can squeeze a bunch (while aligned), the higher the probability for interaction. However, the amount of squeezing is limited by the EM repulsion of the protons and also by the Heisenberg uncertainty relation.

Comparing the cross sections for different processes to each other shows which process is more abundant and by how much. In practice, the processes with the highest cross section is elastic scattering - protons simply bouncing of one another without breaking apart. A diagram of the cross sections for different processes and the corresponding number of events per second at a typical LHC luminosity can be found here: `inspirehep.net/record/780632/plots` .