Tutorial 12

Topics for today

- P1: Weyl fields (and problem of coupling by mass)
- P2: Transformation properties of vectors/axial vectors
- P3: Massive vector bosons (the Proca equation)
- P4: EoM for the Yang-Mills Lagrangian
- P5: SU(2) charges of weak interactions

Problem 1

Consider

$$\begin{aligned}\psi_{\mathrm{L}} &= \frac{1}{2}(1-\gamma_5)\psi,\\ \psi_{\mathrm{R}} &= \frac{1}{2}(1+\gamma_5)\psi,\end{aligned}$$

where ψ is a Dirac spinor. Derive the equations of motion for these fields. Show that they are decoupled in the case of a massless spinor. (These fields ψ_{L} and ψ_{R} are known as Weyl fields.)

Problem 2 ☼

For a Dirac field ψ, obtain the action of Lorentz transformations on ψ. Investigate the transformation properties under proper orthochronous Lorentz transformation of:

- $\bar{\psi}\psi$,
- $\bar{\psi}\gamma^{\mu}\psi$,
- $\bar{\psi}\gamma^{\mu}\gamma^{5}\psi$.

Assumptions

Let Λ be an ocrthochronous Lorentz transformations,

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \qquad \text{MS (A.48)}$$

i.e. $\Lambda^{0}{}_{0} > 0$ and $\det(\Lambda^{\mu}{}_{\nu}) = \pm 1$, so that the sense of time is not reversed, but the transformation may or may not involve spatial inversion.

It can be shown[1] corresponding that to each such Lorentz transformation Λ one can construct a non-singular 4x4 matrix $S = S(\Lambda)$ with the properties

$$\gamma^{\nu} = \Lambda^{\nu}{}_{\mu}S\gamma^{\mu}S^{-1} \qquad \text{MS (A.49)}$$

[1] For its derivation Mandl and Shaw referenced pp. 358-359 in H. A. Bethe and R. W. Jackiw, *Intermediate Quantum Mechanics*, 2nd edn, Benjamin, New York, 1968.

and

$$S^{-1} = \gamma^0 S^\dagger \gamma^0. \qquad \text{MS (A.50)}$$

If the transformation properties of the Dirac spinor $\psi(x)$ are defined by

$$\psi(x) \to \psi'(x') = S\psi(x), \qquad \text{MS (A.52)}$$

we have the transformation properties of the corresponding adjoint spinor

$$\bar{\psi}(x) \to \bar{\psi}'(x') = (S\psi(x))^\dagger \gamma^0 = \psi^\dagger(x) S^\dagger \gamma^0 = \psi^\dagger(x)\gamma^0 S^{-1} = \bar{\psi}(x) S^{-1}.$$

Problem 3, Massive vector bosons

Derive equations of motion for the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_W^2 W^\mu W_\mu$$

and interpret the results.

Problem 4, EoM for the Yang-Mills Lagrangian ☼

The Lagrangian density for SU(n) gauge fields (also called Yang-Mills Lagrangian) reads,

$$\mathcal{L}^{\text{YM}} = -\frac{1}{4}\text{Tr} F^2 == -\frac{1}{4} F^j_{\mu\nu} F^{\mu\nu j}.$$

Evaluate the equation of motion for A^i_μ, expressing it in covariant form.

Use the convention where the generators of the Lie algebra corresponding to the F-quantities are satisfying

$$[T_i, T_j] = \mathrm{i} f_{ijk} T_k, \qquad \text{Tr}(T_i T_j) = \delta_{ij}. \tag{4.1}$$

Problem 5, SU(2) charges of weak interactions

When assigning charges to different object, it is not relevant how many different operators exist in a theory. Rather it is important how many operators can be found that simultaneously commutes. In this course, we are looking at weak interactions, with is a SU(2) theory. As such there are three generators τ_1, τ_2 and τ_3 with the corresponding charges Q_1, Q_2 and Q_3. However, as these generators do not commute $[\tau_i, \tau_j] = \mathrm{i}\epsilon_{ijk}\tau_k$, and neither does the charges$[Q_i, Q_j] = \mathrm{i}\epsilon_{ijk} Q_k$. Using the familiar problem of the spin assignment $J_\pm$, J_z in SU(2), carry out the similar procedure as to get $[\, J_z,\, J_\pm\,] = \pm J_\pm$ to get the currents J^μ, $J^{\dagger\mu}$ that carry weak charges ± 1.