Tutorial 2

Topics for today

- Natural Units and Dimensional Analysis
- Notation, Fourier Transform, Dirac delta function (contd.)
- Ladder operators, Heisenberg algebra

1 Recap: SR and Lorentz Invariance

In the previous tutorial we have addressed the role of SR and the Lorentz invariance. Here is a recap:

Notation: In the following, the Einstein summation convention is used, which says that a sum is understood over any repeated index. Also, symmetrization and anti-symmetrization are often denoted by parentheses and brackets around the indices concerned. For instance, for any tensor $T_{\alpha\beta}$,

$$T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right), \quad T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right). \tag{1.1}$$

The arena of SR is the Minkowski spacetime (usually denoted as $\mathbb{R}^{1,3}$, or sometimes M^4) which is a pseudo-Euclidean space with the Minkowski metric η having the signature,

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \tag{1.2}$$

(There is also an alternative signature $(-, +, +, +)$; the most literature in GR uses the latter while particle physicists tend to use the former.)

The inverse of the metric is,

$$(\eta^{-1})^{\mu\sigma}\eta_{\sigma\nu} = \delta^{\mu}_{\nu}, \tag{1.3}$$

for which we use the same symbol (as there is no confusion),

$$\eta^{\mu\nu} := (\eta^{-1})^{\mu\nu}, \tag{1.4}$$

The inverse of the Minkowski metric has the same components,

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.5}$$

We distinguish two vector spaces, one containing ordinary vectors, identified by the components having the indices up, e.g.,

$$x^{\mu} = (x^0, x^1, x^2, x^3) = (x^0, x^i) = (t, \mathbf{x}), \tag{1.6}$$

$$p^{\mu} = (p^0, p^1, p^2, p^3) = (p^0, p^i) = (E, \mathbf{p}), \tag{1.7}$$

and the corresponding dual vector space with covectors (i.e., linear functionals) that we identify by the components with the indices down, e.g.,

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_0, \partial_i) = (\partial_t, \boldsymbol{\nabla})$$

$$p_{\mu} = (p_0, p_1, p_2, p_3) = (p_0, p_i) = (E, -\mathbf{p}).$$

We use the metric to raise/lower indices, e.g.,

$$p_\mu = \eta_{\mu\nu} p^\nu, \qquad p^\mu = \eta^{\mu\nu} p_\nu. \tag{1.8}$$

The Lorentz transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.9}$$

is the homogeneous isometry of Minkowski spacetime which leave the square of the length of a vector invariant. The matrix Λ representing the transformation is a constant matrix; x^μ and x'^μ are the coordinates of the same event in two different inertial frames. The matrix Λ must satisfy the condition,

$$\eta = \Lambda^{\mathrm{T}} \eta \Lambda, \tag{1.10}$$

implying,

$$\det \Lambda = \pm 1. \tag{1.11}$$

Moreover, we necessarily have (Problem 1 in Tutorial 1),

$$\left| \Lambda^0{}_0 \right| \geq 1. \tag{1.12}$$

The Lorentz transformation can be decomposed into the symmetric part (boost) and the orthogonal part (rotations) which can be continuously parametrized, plus two discrete transformations: time inversion and space reflections. The Lorentz transformation with $\det \Lambda = 1$ are called proper and with $\Lambda^0{}_0 \geq 1 > 0$ are called orthochronous. The orthochronous and proper are also known as the reduced Lorentz transformations.

Under the Lorentz transformation,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad x' = \Lambda x, \quad x = \Lambda^{-1} x', \tag{1.13}$$

the scalar field $\phi(x)$ transform as,

$$\phi(x) \to \phi'(x') = \phi(x) = \phi(\Lambda^{-1} x'), \tag{1.14}$$

the vector field $V^\mu(x)$ transform as,

$$V^\mu(x) \to V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1} x'), \tag{1.15}$$

the covector field $A_\mu(x)$ transform as,

$$A_\mu(x) \to A'_\mu(x') = (\Lambda^{-1,\mathrm{T}})_\mu{}^\nu A_\nu(x) = (\Lambda^{-1,\mathrm{T}})_\mu{}^\nu A_\nu(\Lambda^{-1} x') \tag{1.16}$$

and the Maxwell tensor field $F_{\mu\nu}(x)$ transforms accordingly,

$$F_{\mu\nu}(x) \to F'_{\mu\nu}(x') = (\Lambda^{-1,\mathrm{T}})_\mu{}^\rho F_{\rho\sigma}(x) (\Lambda^{-1})^\sigma{}_\nu \tag{1.17}$$

$$= (\Lambda^{-1,\mathrm{T}})_\mu{}^\rho F_{\rho\sigma}(\Lambda^{-1} x') (\Lambda^{-1})^\sigma{}_\nu. \tag{1.18}$$

2 Natural Units and Dimensional Analysis

In QFT, expressions and calculations are much simplified if one uses natural units. In natural units one takes mass, action and velocity as fundamental dimensions and chooses the velocity of light c as unit of velocity and the Planck constant $\hbar$ as unit of action. Hence,

$$c = \hbar = 1, \tag{2.1}$$

In the SI system we have

$$[c]_{\text{SI}} = \mathsf{L}\,\mathsf{T}^{-1}, \qquad [\hbar]_{\text{SI}} = \mathsf{M}\,\mathsf{L}^2\,\mathsf{T}^{-1}, \tag{2.2}$$

which becomes,

$$[c]_{\text{n.u.}} = 1, \qquad [\hbar]_{\text{n.u.}} = 1. \tag{2.3}$$

Hence, in n.u. we have,

$$\mathsf{L} = \mathsf{T} = \mathsf{M}^{-1}, \tag{2.4}$$

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}, \tag{2.5}$$

and enables us to express all quantities in terms of energy (or mass):

$$[\mathrm{d}x] = [x] = [t] = \mathsf{M}^{-1}, \tag{2.6}$$

$$[\partial_\mu] = [p^\mu] = \mathsf{M}, \tag{2.7}$$

$$[\mathrm{d}^4x] = \mathsf{M}^{-4}, \qquad [S] = 1, \qquad [\mathcal{L}] = \mathsf{M}^4. \tag{2.8}$$

One can always recover $\hbar$ and c factors by dimensional analysis.
See Problem 1.

3 Recap: Fourier transform, Notation; Dirac delta function

The Fourier transform has many forms, where in physics, we mostly use the one written in terms of angular frequency ω,

$$\hat{f}(\omega) = \int_{\mathbb{R}^n} \mathrm{d}^n x\, f(x)\,\mathrm{e}^{-\mathrm{i}\omega\cdot x}. \tag{3.1}$$

Under this convention, the inverse transform becomes,

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{d}^n\omega\, \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega\cdot x}. \tag{3.2}$$

When the Fourier transform is defined this way, it is no longer a unitary transformation on $L^2(\mathbb{R}^n)$. There is also less symmetry between the formulae for the Fourier transform and its inverse. Another convention is to split the factor of $(2\pi)^n$ evenly between the Fourier transform and its inverse, which leads to definitions (convention used in Sakurai, e.g.),

$$\hat{f}(\omega) = \frac{1}{(2\pi)^{n/2}}\int \mathrm{d}^n x\, f(x)\,\mathrm{e}^{-\mathrm{i}\omega\cdot x}, \quad f(x) = \frac{1}{(2\pi)^{n/2}}\int \mathrm{d}^n\omega\, \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega\cdot x}. \tag{3.3}$$

Fourier transform in braket notation. Sakurai & Napolitano - *Modern Quantum Mechanics*

$$\langle \mathbf{x}'|\alpha\rangle = \psi_\alpha(\mathbf{x}'), \qquad \langle \mathbf{p}'|\alpha\rangle = \phi_\alpha(\mathbf{p}')$$
$$\int \mathrm{d}^3\mathbf{x}'\, |\mathbf{x}'\rangle\langle \mathbf{x}'| = \mathbb{1}, \quad \langle \mathbf{x}'|\mathbf{x}''\rangle = \delta^3(\mathbf{x}'-\mathbf{x}'')$$
$$\int \mathrm{d}^3\mathbf{p}'\, |\mathbf{p}'\rangle\langle \mathbf{p}'| = \mathbb{1}, \quad \langle \mathbf{p}'|\mathbf{p}''\rangle = \delta^3(\mathbf{p}'-\mathbf{p}'')$$
$$\langle \mathbf{x}'|\mathbf{p}'\rangle = \frac{1}{(2\pi\hbar)^{3/2}} \cdot \exp\left(\mathrm{i}\mathbf{p}'\cdot\mathbf{x}'/\hbar\right)$$
$$\langle \mathbf{p}'|\mathbf{x}'\rangle = \langle \mathbf{x}'|\mathbf{p}'\rangle^* = \frac{1}{(2\pi\hbar)^{3/2}} \cdot \exp\left(-\mathrm{i}\mathbf{p}'\cdot\mathbf{x}'/\hbar\right)$$
$$\text{cf. discrete: } \sum_n |n\rangle\langle n| = \mathbb{1}, \quad \langle n|m\rangle = \delta_{nm}, \quad \langle n|\alpha\rangle = u_n$$

Now, compare the plane wave solutions $\phi(x)$ for the real Klein-Gordon-Fock equation,

$$(\Box + m^2)\phi(x) = (\partial_\mu\partial^\mu + m^2)\phi(x) = 0, \quad \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2, \tag{3.4}$$

derived as the equations of motion from the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi(x)\right)\left(\partial^\mu\phi(x)\right) - \frac{1}{2}m^2\phi^2(x). \tag{3.5}$$

- Mandl and Shaw, Eq. (3.7)
  (N.b. $\phi(x)$ is an operator , as well as a, $a^\dagger$!)

$$\phi(x) = \phi^+(x) + \phi^-(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} \left(a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right) \tag{3.6}$$
$$\phi^+(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} \tag{3.7}$$
$$\phi^+(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx} \tag{3.8}$$

- Schwartz, Eq. (2.78)

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}px} + a_{\mathbf{p}}^\dagger\mathrm{e}^{\mathrm{i}px}\right) \tag{3.9}$$

- Peskin and Schroeder Eq. (2.47)

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a_{\mathbf{p}}^\dagger\mathrm{e}^{\mathrm{i}p\cdot x}\right)|_{p^0=E_{\mathbf{p}}} \tag{3.10}$$

Dirac Delta Function, properties. Jackson - *Classical Electrodynamics*, 3rd ed
The Dirac delta function, or δ function, is a generalized function, or distribution, on the real number line that is zero everywhere except at zero, with an integral of one over the entire real

line. It has many representations, one the most common is given in (7°) below.

$$1^\circ(\text{definition}) \int \mathrm{d}x\, \delta(x)\, f(x) = f(0), \quad (\text{i.e. } \langle \delta, f\rangle = f(0))$$

$$2^\circ\ \delta(-x) = \delta(x), \quad 3^\circ\ \delta(ax) = \frac{\delta(x)}{|a|}, \quad 4^\circ \int \mathrm{d}x\, f(x)\delta(x-a) = f(a),$$

$$5^\circ\ \delta(g(x)) = \sum_{x_i \in g^{-1}(\{0\})} \frac{\delta(x-x_i)}{\left|\frac{\partial g(x)}{\partial x}\right|_{x=x_i}}, \quad x_i \in g^{-1}(\{0\}) = \{\, x \,|\, g(x) = 0\,\},$$

$$6^\circ\ \delta^3(\mathbf{x}) = \delta(x)\delta(y)\delta(z), \quad 7^\circ\ \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}xk}$$

Questions: Is $(\delta(\mathbf{x}))^3$ valid? What does mean $\delta^3(\mathbf{x})$? What about $\delta^{(3)}(\mathbf{x})$? What is dimension of $\delta^3(\mathbf{x})$, if $\mathbf{x}$ is in meters? (Hint: for the point electrical charge distribution we have: $\rho(\mathbf{x}) = \sum_i Q_i \delta^3(\mathbf{x} - \mathbf{x}_i)$.)

One of the most useful properties of the delta function is the composition with a function (Property 5° above),

$$\delta(g(x)) = \sum_{\text{zeros } f(x_0)=0} \frac{\delta(x-x_0)}{g'(x)|_{x=x_0}}, \tag{3.11}$$

Later on in the course, this expression will be used, e.g., in deriving expressions for the differential cross-sections. See Mandl and Shaw Eqs. (8.11) and (8.14) used to derive (8.15). Compare the following expressions,

- Schwartz, Problem (2.6) // cf. Mandl and Shaw Eq. (8.11)

$$\int \mathrm{d}k^0 \delta(k^2 - m^2)\theta(k^0) = \frac{1}{2\omega_{\mathbf{k}}}, \qquad \omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} > 0, \tag{3.12}$$

- Peskin & Schroeder Eq. (2.40)

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{E_{\mathbf{p}}} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\big|_{p^0>0}, \qquad E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2} > 0, \tag{3.13}$$

- Srednicki, Eq. (3.16)

$$\int \mathrm{d}k^0 \delta(k^2 + m^2)\theta(k^0) = \frac{1}{2\omega}, \qquad \omega = \sqrt{\mathbf{k}^2 + m^2} > 0, \tag{3.14}$$

- Reg Fourier transform and Lorentz invariant form, Schwartz, Eq. (2.72) // cf. Sakurai

$$\langle \mathbf{p}|\mathbf{k}\rangle = 2\omega_{\mathbf{k}}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{k}), \tag{3.15}$$

$$\mathbb{1} = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}} |\mathbf{p}\rangle\langle\mathbf{p}| \tag{3.16}$$

4 Ladder operators and Heisenberg Algebra

Useful Commutator Identities

$$[\,A,\, BC\,] = [\,A, B\,]\,C + [\,B, A\,]\,C, \tag{4.1}$$

$$[\,AB,\, C\,] = A\,[\,B, C\,] + [\,A, C\,]\,B. \tag{4.2}$$

Quantum Harmonic Oscillator, reminder

(See also Section 1.2.2 Harmonic oscillator in Mandl and Shaw.)

We start from the Hamiltonian of the QHO,

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \tag{4.3}$$

Now, we normalize the Hamiltonian $H' = H/\omega$ redefining the variables

$$x' \to x\sqrt{m\omega} \tag{4.4}$$

$$p' \to p/\sqrt{m\omega} \tag{4.5}$$

so that,

$$\frac{H}{\omega} = \frac{1}{2}\left(\frac{p}{\sqrt{m\omega}}\right)^2 + \frac{1}{2}\left(x\sqrt{m\omega}\right)^2, \quad H' = \frac{1}{2}p'^2 + \frac{1}{2}x'^2. \tag{4.6}$$

Since we have the canonical commutation relation,

$$\boxed{[x', p'] = [x, p] = \mathrm{i}}, \tag{4.7}$$

the Hamiltonian can not be diagonalized directly. We notice however that H is quadratic, therefore we should always be able to rewrite it on a form that is diagonal.

N.B. For the simplicity, from now on we continue with x' and p' without the primes!

It is easy to verify that by defining the ladder operators,

$$a = \frac{1}{\sqrt{2}}\left[x + \mathrm{i}p\right], \qquad \text{(the anihilation operator)} \tag{4.8}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left[x - \mathrm{i}p\right], \qquad \text{(the creation operator)} \tag{4.9}$$

we can complete the squares and obtain,

$$a^\dagger a = \frac{1}{\sqrt{2}}\left[x - \mathrm{i}p\right]\frac{1}{\sqrt{2}}\left[x + \mathrm{i}p\right] \tag{4.10}$$

$$= \frac{1}{2}\left[x^2 + p^2 + \mathrm{i}xp - \mathrm{i}px\right] \tag{4.11}$$

$$= \frac{1}{2}\left[x^2 + p^2 + \mathrm{i}[x, p]\right] \tag{4.12}$$

$$= \frac{1}{2}\left[x^2 + p^2\right] - \frac{1}{2}, \tag{4.13}$$

so that,

$$H = a^\dagger a + \frac{1}{2}, \tag{4.14}$$

or in the original unprimed variables $H = \omega\left(a^\dagger a + \frac{1}{2}\right)$.

It can be also verified that,

$$[a, a^\dagger] = \left[\frac{1}{\sqrt{2}}\left[x + \mathrm{i}p\right], \frac{1}{\sqrt{2}}\left[x - \mathrm{i}p\right]\right] = \frac{1}{2}\left[x, -\mathrm{i}p\right] + \frac{1}{2}\left[-\mathrm{i}p, x\right] = 1, \tag{4.15}$$

that is,

$$\boxed{[a, a^\dagger] = 1, \qquad aa^\dagger = 1 + a^\dagger a}. \tag{4.16}$$

This is an important property which is used to define the **Heisenberg algebra**.

We begin by introducing the number operator N,

$$N := a^\dagger a, \qquad \Rightarrow \qquad N = N^\dagger, \tag{4.17}$$

which means that N has real eigenvalues. Let λ be the eigenvalue for an eigenstate $|\lambda\rangle$,

$$N |\lambda\rangle = \lambda |\lambda\rangle , \tag{4.18}$$

then for $|\lambda'\rangle = a |\lambda\rangle$,

$$\langle\lambda|N|\lambda\rangle = \left\langle \lambda \middle| a^\dagger a \middle| \lambda \right\rangle = \lambda^2 \langle\lambda|\lambda\rangle = \langle\lambda'|\lambda'\rangle \geq 0. \tag{4.19}$$

thus the spectra of N is non-negative.

We have also,

$$[N, a] = [a^\dagger a, a] = a^\dagger [a, a] - a[a^\dagger, a] = a[a, a^\dagger] = -a, \tag{4.20}$$

$$[N, a^\dagger] = [a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] - a[a^\dagger, a^\dagger] = a^\dagger [a, a^\dagger] = a^\dagger, \tag{4.21}$$

$$[N, a] = -a, \qquad [N, a^\dagger] == a^\dagger. \tag{4.22}$$

Starting from $N |\lambda\rangle = \lambda |\lambda\rangle$ and the above relations,

$$N \left(a^\dagger |\lambda\rangle\right) = (\lambda + 1) |\lambda\rangle , \tag{4.23}$$

$$N (a |\lambda\rangle) = (\lambda - 1) |\lambda\rangle , \tag{4.24}$$

which means that a^n applied any eigenstate $|\lambda\rangle$ eventually becomes negative, so the maximum n is $n = \lfloor\lambda\rfloor$

$$a^{\lfloor\lambda\rfloor} |\lambda\rangle = (\lambda - \lfloor\lambda\rfloor) |\lambda\rangle , \tag{4.25}$$

otherwise, if we have one more a applied, the state will be with the negative eigenvalue. Thus we impose

$$a |\lambda - \lfloor\lambda\rfloor\rangle = 0, \tag{4.26}$$

to avoid the contradiction with negative eigenvalues (to flip-over from positive to negative; this will stop down-counting to eigenstate with eigenvalue 0). Hence we always have state $|0\rangle := |\lambda - \lfloor\lambda\rfloor\rangle$ with eigenvalue 0,

$$N |0\rangle = 0 |0\rangle , \qquad a |0\rangle = 0. \tag{4.27}$$

All other states can be generated by using $a^\dagger$ as $(a^\dagger)^n |0\rangle$. We want states to be normalized to 1, so we define,

$$\langle 0|0\rangle := 1, \tag{4.28}$$

which gives

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |n\rangle . \tag{4.29}$$

5 Problems

Problem 1. Assume that, during the calculation of the cross-section in natural units, we have arrived at the equation $\sigma \propto \frac{1}{E^2}$, where E is GeV. Recover $\hbar$ and c factors by dimensional analysis to express the equation in the SI units.

Problem 2.

1. Evaluate,

$$[a^\dagger a, a], \qquad [a^\dagger a, a^\dagger], \qquad [a, (a^\dagger)^n]. \tag{5.1}$$

2. Prove,

$$[\, a,\ f(a^\dagger)\,] = \frac{\partial}{\partial a^\dagger} f(a^\dagger), \qquad [\, f(a),\ a^\dagger\,] = \frac{\partial}{\partial a} f(a) \tag{5.2}$$

Problem 3. First show that,

$$a\, f(a^\dagger)\, |0\rangle = [a, f(a^\dagger)]\, |0\rangle\,, \tag{5.3}$$

$$a^n\, f(a^\dagger)\, |0\rangle = \frac{\partial^n}{\partial (a^\dagger)^n} f(a^\dagger)\, |0\rangle\,, \tag{5.4}$$

then calculate the following expectation values,

$$\left\langle 0 \middle| a(a^\dagger)^2 \middle| 0 \right\rangle, \qquad \left\langle 1 \middle| a^\dagger a a^\dagger \middle| 0 \right\rangle, \qquad \langle n|x|n\rangle\,, \qquad \left\langle n \middle| x^2 \middle| n \right\rangle, \qquad \left\langle 0 \middle| a^n \mathrm{e}^{\alpha a^\dagger} \middle| 0 \right\rangle. \tag{5.5}$$