Tutorial 4

FK8027 - Quantum Field Theory

Monday 20^{th} November, 2017

Topics for today

- Invariance in form and value
- Rotations in 3D euclidean space and the definition of a group
- The Lorentz group
- The unitary group
- The generators of a group of differentiable transformations

1 Invariance in form and value

An important concept which is related with the "symmetries of a system" is the concept of invariance. If we have a system which is invariant under a certain transformation, then we say that the transformation is a symmetry for that system. Being invariant means that the system stays the same, is equal to itself before and after the transformation.

At this point, it is important to emphasize the difference between two types of invariance that we can encounter: invariance "in form" and invariance "in value". We use the scalar product of two vectors as an example.

The scalar product is invariant in value under generic coordinate transformations (GCT). We know that from basic geometry. This means the following. Take two vectors written in certain coordinates, and compute the scalar product between them. You get the value a. Now do a GCT; the vectors will transform (contravariantly) and their components will be different. However, if you compute their scalar product in the new coordinate system, you get a again. This is invariance in value. However, this does not imply invariance in form, which is a stronger statement. Invariance in form means that the form of the mathematical expressions stays the same before and after the transformation (i.e., the transformation is a symmetry). In euclidean space and in cartesian coordinates, we know that the scalar product between v^μ and u^μ is

$$v^\mu u_\mu = v^1 u_1 + v^2 u_2 + v^3 u_3. \tag{1}$$

If we change to spherical polar coordinates, the vectors have coordinates (r_i, θ_i, ϕ_i) and the expression (the form) for the scalar product is different (please check)

$$v^\mu u_\mu = r_v\, r_u \left[\sin(\phi_v)\sin(\phi_u)\cos(\theta_v - \theta_u) + \cos(\phi_v)\cos(\phi_u)\right], \tag{2}$$

which is not the sum of the products of the components anymore. Remember that covariant and contravariant indices are interchangeable in euclidean space. In this case, being invariant in form means that the expression for the scalar product is the same even for the transformed vectors, but with the new components.

In the following, we will be interested in transformations that keep physical quantities invariant in form and value.

2 The orthogonal group of euclidean rotations

We have seen that, in the relativistic notation (contravariant and covariant indices), the scalar product between two vectors v^μ and u^μ can be written as

$$\vec{u} \cdot \vec{v} = u_\mu v^\mu = g_{\mu\nu} u^\mu v^\mu. \tag{3}$$

Then the squared norm of a vector can be defined as

$$\vec{v} \cdot \vec{v} = g_{\mu\nu} v^\mu v^\mu. \tag{4}$$

In the familiar 3-dimensional euclidean space, we have

$$g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

i.e. the metric is the identity ($\mathbb{1}$, in matrix notation). The scalar product is then,

$$v_\mu u^\mu = v_1 u^1 + v_2 u^2 + v_3 u^3 = v^1 u^1 + v^2 u^2 + v^3 u^3. \tag{6}$$

We know that the squared norm (from now on, we call it the norm) of a vector does not change in form and value under a spatial rotation described by the matrix $R^\mu{}_\nu$.[1] We can write this in the relativistic notation,

$$v'^\mu = R^\mu{}_\nu v^\nu, \qquad \delta_{\mu\nu} v^\mu v^\nu = \delta_{\mu\nu} v'^\mu v'^\nu = \delta_{\mu\nu} R^\mu{}_\rho v^\rho R^\nu{}_\sigma v^\sigma, \tag{7}$$

which can be rewritten as

$$\delta_{\rho\sigma} v^\rho v^\sigma = \left[(R^\intercal)_\rho{}^\mu \delta_{\mu\nu} R^\nu{}_\sigma \right] v^\rho v^\sigma. \tag{8}$$

This must hold for every vector v^μ, therefore we have

$$\delta_{\rho\sigma} = (R^\intercal)_\rho{}^\mu \delta_{\mu\nu} R^\nu{}_\sigma. \tag{9}$$

[1]The scalar product depends on the moduli of the two vectors and the relative angle between them, which does not change if we rotate both vectors by the same angle.

In matrix notation

$$\mathbb{1} = R^\mathsf{T}\, \mathbb{1}\, R. \tag{10}$$

Note that this implies $R^\mathsf{T} = R^{-1}$. The usual way to introduce spatial rotations is to *define* the matrix R through (10). A matrix fulfilling (10) is said to be "orthogonal", and the set of all possible orthogonal matrices of dimension $n \times n$ is called $O(n)$. In our case, the dimension of the space is three and then the set is called $O(3)$. If we take the determinant of both sides of (10), we get

$$\det(\mathbb{1}) = \det(R^\mathsf{T}\, \mathbb{1}\, R) = \det(R^\mathsf{T}) \det(\mathbb{1}) \det(R)\,. \tag{11}$$

Since $\det(R^\mathsf{T}) = \det(R)$, this implies,

$$\det(R) = \pm 1. \tag{12}$$

If the determinant is $+1$, we are considering proper rotations (i.e., continuous rotations); if the determinant is -1, we are considering reflections, which are not continuous, but discrete transformations. The set of all matrices satisfying (10) and having determinant $+1$ is called $SO(n)$ in n-dimensions, that is "special orthogonal".

The set $O(n)$ is a "group", a concept which we are going to define now.

Definition. A group is a non-empty set G, together with an operation $\circ$ (called the "group law") which takes two generic elements g_1, g_2 of G and associates another element $g_1 \circ g_2$ to them. The group law must be such that the following properties hold:

- **Closure.** $g_1 \circ g_2 \in G, \quad \forall\, g_1, g_2 \in G.$
- **Associativity.** $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3, \quad \forall\, g_1, g_2 \in G.$
- **Identity element.** There exist an element e such that

  $$g \circ e = e \circ g = g, \quad \forall\, g \in G.$$

  e is called the "identity element" of the group, and it can be shown that, if it exists, it is *unique*.
- **Inverse element.** For each $g \in G$, there exist an element $h \in G$, commonly denoted with g^{-1} or $-g$, such that $g \circ h = h \circ g = e$.

These four properties are called the "group axioms". Note that we are *not* guaranteed that the group law is commutative generically, i.e.

$$g_1 \circ g_2 \neq g_2 \circ g_1. \tag{13}$$

If the group law is commutative, then the group is called "abelian"; otherwise, it is called "non–abelian". A simple example of an abelian group is the real line $\mathbb{R}$ with the addition $+$ as the group law (please check).

As we can easily accept, the set of orthogonal matrices $O(3)$ with the matrix product as the group law, is a group. If we apply two consecutive improper rotations (meaning rotations or reflections) R and T to the space, the result of this operation is another improper rotation S. In formulas,

$$T \cdot R = S, \tag{14}$$

where $\cdot$ denotes the matrix product. This means that $O(3)$ is closed. It is associative because the matrix product is such. The identity element is simply the rotation by a zero angle. The inverse element of a rotation by θ is simply the rotation by $-\theta$, and the inverse element of a reflection is the opposite reflection. Therefore, from now on, we will talk about the $O(3)$ orthogonal *group*. $SO(3)$ is also a group, and in particular it is a "subgroup" of $O(3)$.

Definition. A subgroup H of a group G is a subset of G which is a group under the same group law of G.

Therefore, we will also talk about the special orthogonal group, which is the group of proper (i.e., continuous) rotations. The orthogonal group $O(3)$ and the special orthogonal group $SO(3)$ are very important in classical mechanics. All mechanical laws have to be invariant in form under rotations, and this can be mathematically stated by saying that the physical laws have to be invariant under a transformation belonging to $O(3)$.[2] When physical laws are invariant in form and value under a given transformation, this transformation is called a "symmetry" of the system. We can then say that spatial rotations are symmetries for classical mechanics.

3 The Lorentz group

In QFT, we use special relativity and therefore the metric is not the identity. It is the Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{15}$$

[2]For the more interested people, the laws of classical mechanics have to be invariant in form and value also under translations and uniform motions, i.e. they have to be same in any inertial frame; look for "Galilean group".

as we already know. Therefore, the scalar product is different (remember that we are in four dimensions now)

$$\vec{v} \cdot \vec{u} = v_\mu u^\mu = \eta_{\mu\nu} v^\mu u^\mu = v^0 u^0 - v^1 u^1 - v^2 u^2 - v^3 u^3 \\ = v_0 u^0 + v_1 u^1 + v_2 u^2 + v_3 u^3. \tag{16}$$

We are interested in finding the symmetries of special relativity. Again, we can *define* them as those symmetries which preserve the scalar product (16). We then consider a transformation $L^\mu{}_\nu$ acting on a four-vector v^μ, and we *require* that the Minkowski scalar product is kept invariant in form and value,

$$v'^\mu = L^\mu{}_\nu v^\nu, \qquad \eta_{\mu\nu} v^\mu v^\nu = \eta_{\mu\nu} v'^\mu v'^\nu = \eta_{\mu\nu} L^\mu{}_\rho v^\rho L^\nu{}_\sigma v^\sigma, \tag{17}$$

Similarly to the euclidean case, we can write

$$\eta_{\rho\sigma} = \left(L^\intercal\right)_\rho{}^\mu \eta_{\mu\nu} L^\nu{}_\sigma, \tag{18}$$

or, in matrix notation,

$$\eta = L^\intercal \, \eta \, L. \tag{19}$$

The equation (19) is the *definition* of "Lorentz transformations". They are transformations preserving the Minkowski scalar product or, equivalently, the Minkowski metric. If we multiply both sides by -1, then $\eta_{\mu\nu} \to -\eta_{\mu\nu}$ and $L^\mu{}_\nu$ is unchanged. This shows that the signature is not important for our purposes. An analogous calculation as the one done for $O(3)$ shows us that

$$\det(L) = \pm 1. \tag{20}$$

These transformations also form a group, called the "Lorentz group" $O(1,3)$, i.e. the "pseudo–orthogonal" group for a metric with signature $(+,-,-,-)$ [or $(-,+,+,+)$]. We studied some properties of this group also in previous tutorials (see Handouts for Tutorial 1 and 2). The Lorentz group is then the symmetry group of special relativity (and QFT).

4 The unitary group

So far, we have introduced two real groups. Now we talk about a complex group. We define a "unitary" matrix U as a matrix satisfying

$$U^\dagger \, \mathbb{1} \, U = U \, \mathbb{1} \, U^\dagger = \mathbb{1}, \tag{21}$$

where $\dagger$ means "transpose conjugate". Then we immediately see that, when we apply U to a complex vector z (i.e., a vector with complex components;

in quantum mechanics this vector is a ket, and its adjoint is a bra), we get (in matrix notation)

$$z' = Uz, \qquad z'^{\dagger} z' = (Uz)^{\dagger}\, Uz = z^{\dagger} U^{\dagger} U z = z^{\dagger} z, \tag{22}$$

so the unitary matrices preserve in form and value the inner product of quantum mechanics between bras and kets (just replace z by $|\lambda\rangle$ and $z^{\dagger}$ with $\langle\lambda|$). The set of unitary matrices form another group, called the unitary group $U(n)$ in n *complex* dimensions. From (21) we obtain,

$$U^{\dagger} = U^{-1}, \qquad |\det(U)| = 1 \Rightarrow \det(U) = \mathrm{e}^{\mathrm{i}\phi}. \tag{23}$$

Again, if we impose $\det(U) = 1$, then we have a subgroup of the unitary group, called the “special unitary” group $SU(n)$.

We now compute the number of independent *real* components of a unitary matrix. A $n \times n$ complex matrix has n^2 complex components, i.e. $2n^2$ real components (real and imaginary part). If we impose that it is unitary, we have to impose the equations in (21). A generic $n \times n$ complex matrix can be written as,

$$U = \begin{pmatrix} a & \dots & b \\ \vdots & \ddots & \vdots \\ c & \dots & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C}. \tag{24}$$

Then (21) reads

$$UU^{\dagger} = \begin{pmatrix} a & \dots & b \\ \vdots & \ddots & \vdots \\ c & \dots & d \end{pmatrix} \cdot \begin{pmatrix} a^* & \dots & c^* \\ \vdots & \ddots & \vdots \\ b^* & \dots & d^* \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix} = \mathbb{1}, \tag{25}$$

which is equal to

$$\begin{pmatrix} |a|^2 + \dots + |b|^2 & \dots & a\,c^* + \dots + b\,d^* \\ \vdots & \ddots & \vdots \\ c\,a^* + \dots + d\,b^* & \dots & |c|^2 + \dots + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}. \tag{26}$$

The diagonal equations are real, so they count as n equations. The non-diagonal equations are complex, so they count as two equations each (the real and the imaginary part have to be zero separately). However, we can see that the equation (ji) is the conjugate of the equation (ij), therefore we have to consider only the upper triangle. Hence we have d independent equations, where d is

$$d = \#^{\text{eqs}}_{\text{real,diagonal}} + \#^{\text{eqs}}_{\text{real,nondiagonal}} = n + \left(\frac{n^2 - n}{2}\right) 2 = n^2. \tag{27}$$

Therefore, the number of independent components of a unitary matrix is

$$2n^2 - d = 2n^2 - n^2 = n^2. \tag{28}$$

For a matrix in $SU(n)$, the independent components are $n^2 - 1$, because we also impose one more equation, namely that the determinant is 1.

For $n = 2$, a matrix in $U(2)$ has 4 independent components, or "degrees of freedom", whereas a matrix in $SU(2)$ has 3 degrees of freedom.

5 The generators of a group of differentiable transformations

The concepts of generators of a group has a deep meaning and can be associated with every group whose elements are differentiable transformations [for example, to $O(n), O(n, m), U(n)$]. We will talk about the generators of $SU(2)$, but it is important to know that an analogous analysis can be carried out for other groups too, *mutatis mutandis*.

We saw that a matrix in $SU(2)$ has three degrees of freedom. Naïvely, one would then think that it can be expressed in a basis of matrices having three basis elements. However, $SU(2)$ is a group and not a vector space, so we have to work a bit more. For matrix groups of differentiable transformations, we can write a transformation g as

$$g = \mathrm{e}^{\mathrm{i}\,T}, \tag{29}$$

For $SU(2)$, we can write

$$U = \mathrm{e}^{\mathrm{i}\,H}. \tag{30}$$

We know that this U has to satisfy (21), so

$$UU^\dagger = \mathrm{e}^{\mathrm{i}\,H}\,\mathrm{e}^{-\mathrm{i}\,H^\dagger} = \mathrm{e}^{\mathrm{i}\left(H - H^\dagger\right)} = \mathbb{1}, \tag{31}$$

which implies

$$H = H^\dagger, \tag{32}$$

i.e., H is a hermitian matrix. So a unitary matrix can be expressed as the exponential of a hermitian matrix. A special unitary matrix must have determinant 1, which means

$$1 = \det\left(U\right) = \det\left(\mathrm{e}^{\mathrm{i}\,H}\right) = \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}(H)} \Longrightarrow \mathrm{Tr}(H) = 0, \tag{33}$$

where we used the Jacobi's formula for the determinant of the exponential of a matrix.

It turns out that the set of hermitian matrices with the matrix sum is a vector space over the complex field (the same if they are also traceless). Therefore, we can express any hermitian matrix as

$$H = \theta_1 H_1 + \theta_2 H_2 + \theta_3 H_3, \tag{34}$$

where the θ_i are *complex parameters which can depend on the coordinates x^μ or not.* Therefore, a generic element of $SU(2)$ is

$$U = \mathrm{e}^{\mathrm{i}(\theta_1 H_1 + \theta_2 H_2 + \theta_3 H_3)} \tag{35}$$

Varying the θ_i means varying H, hence varying the group element U, i.e. the transformation. The three matrices H_i are called the "generators" of $SU(2)$ (in general, the generators of the group one is considering). They are three for $SU(2)$ because it has three degrees of freedom.

The generators of $SU(2)$ are then three complex 2-dimensional hermitian traceless linearly independent matrices. These are the Pauli matrices,

$$\sigma_1 \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{36}$$

For the continuous transformations of $U(2)$, everything is the same, but we do not impose anymore the determinant to be 1, and so we have four independent components and we need one more matrix in our basis. The new matrix is simply the 2-dimensional identity, which is linearly independent from the Pauli matrices and it is not traceless.