Tutorial 5 (PRELIMINARY)

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Topics for today

- Complex Scalar Field; Symmetries and Conservation Laws; Noether's theorem
- Representation of the Group, Group Parameters, Lie Groups
- The Rotation Group, SO(N); Unitary Groups, SU(N)

1 Symmetries and Conservation Laws, recap

If $f(x)$ is a function and $F[f(x)]$ is a functional, the *functional derivative*, $\delta F/\delta f$ is defined by the relation

$$\delta F[\phi(x)] = \left(F[\phi+\delta\phi] - F[\phi]\right)(x) = \int \mathrm{d}^4y \boxed{\frac{\delta F[\phi(x)]}{\delta\phi(x)}} \delta\phi(y). \qquad \text{MS (12.53)}$$

where δF is a variation of the functional.

The *action* is given by

$$S = \int \mathrm{d}^4x\, \mathcal{L}(\phi_a, \partial_\mu\phi_a), \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density. From $\delta S = 0$, we get the *Euler-Lagrange equations of motion*

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \tag{1.2}$$

The *cannonical momentum* conjugate to the field variable $\phi_a(x)$ is

$$\pi^a(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_a)} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}. \tag{1.3}$$

The *canonical Hamiltonian density* $\mathcal{H}$ and the Hamiltonian H are

$$\mathcal{H} = \dot{\phi}_a\pi^a - \mathcal{L}, \qquad H = \int \mathrm{d}^3x\mathcal{H} = \int \mathrm{d}^3x \left(\dot{\phi}_a\pi^a - \mathcal{L}\right). \tag{1.4}$$

Noether's theorem
states that if the action is invariant with respect to the continous infinitesimal transformations[1]

$$x_\mu \to x'_\mu = x_\mu + \delta_\epsilon x_\mu, \qquad \delta_\epsilon x^\mu = x'^\mu - x^\mu, \tag{1.5}$$

$$\phi_a(x) \to \phi'_a(x') = \phi_a(x) + \delta_\epsilon\phi_a(x), \qquad \delta_\epsilon\phi_a(x) = \phi'_a(x') - \phi_a(x), \tag{1.6}$$

$$\bar{\delta}_\epsilon\phi_a(x) = \phi'_a(x) - \phi_a(x), \tag{1.7}$$

[1] The index ϵ is related to a symmetry group.
Note that $\bar{\delta}_\epsilon\phi_a(x)$ is calculated with both ϕ' and ϕ having the same argument x.

then the divergence of the *Noether current* J^μ_ϵ is equal to zero $\partial_\mu J^\mu_\epsilon = 0$,

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L}\delta_\epsilon x^\mu = \sum_{r=1}^{d} \epsilon_r J^\mu_{\epsilon_r}, \tag{1.8}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_\epsilon \phi_a - T^\mu{}_\nu \delta_\epsilon x^\nu, \tag{1.9}$$

where $T^\mu{}_\nu$ is the *energy-momentum tensor*

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}\,. \tag{1.10}$$

The *Noether charges* Q_ϵ are constants of motion given by

$$Q_\epsilon = \int \mathrm{d}^3x\, J^0_\epsilon. \tag{1.11}$$

2 Groups, recap

Group theory is related to symmetries and the resulting conservations and, as such, plays a fundamental role in modern particle physics. For example, an important group in physics is the rotation group, which is related to the fact that the laws of physics don't change if you rotate your frame of reference. In general what we are after is a set of equations, or laws of physics, that keep the same mathematical form under various transformations. We are often interested in what is called a representation of the group. Let's denote a representation by F. A representation is a mapping that takes group elements $g \in G$ of a group into linear operators F that preserve the composition rule of the group in the sense that $F(a)F(b) = F(ab)$.

Group Parameters. A group can be a function of one or more parameters. Let a group G be such that individual elements $g \in G$ are specified by a finite set of parameters, say n of them. If we denote the set of parameters by $\{\theta_1, \theta_2, ...\theta_n\}$. The group element is then written as $g = G(\theta_1, \theta_2, ...\theta_n)$. The identity is the group element where the parameters are all set to 0., that is $e = G(0, 0, ..., 0)$

Lie Groups. While there are discrete groups with a finite number of elements, most of the groups we will be concerned with have an infinite number of elements. However, they have a finite set of *continuously varying parameters*. In the expression $g = G(\theta_1, \theta_2, ...\theta_n)$ we have suggestively labeled the parameters as angles, since several important groups in physics are related to rotations. The angles vary continuously over a finite range $0...2\pi$. In addition, the group is parameterized by a finite number of parameters, the angles of rotation. So, if a group G: (i) depends on a finite set of continuous parameters θ_i, and (ii) derivatives of the group elements with respect to all the parameters exist, then we call the group a Lie group (a Lie group is a group that is also a differentiable manifold).

By taking derivatives with respect to the parameters and evaluating the derivative at $\theta_i = 0$, we obtain the generators of the group. If there are n parameters of the group, then there will be n generators such that each generator is given by

$$X_i = \frac{\partial g}{\partial \theta_i}|_{\theta_i=0}.$$

Rotations have a special property, in that they are length preserving (that is, rotate a vector and it maintains the same length). A rotation by $-\theta$ undoes a rotation by θ, hence rotations have an orthogonal or unitary representation. In the case of quantum theory, we seek a unitary representation of the group and choose the generators X_i to be Hermitian. In this case

$$X_i = -\mathrm{i}\frac{\partial g}{\partial \theta_i}|_{\theta_i=0}.$$

For some finite θ, the generators allow us to define a representation of the group. Consider a small real number $\epsilon \to 0$ and use a Taylor expansion to form a representation of the group (which we denote by D)

$$D(\epsilon\theta) \approx 1 + \mathrm{i}\epsilon\theta X.$$

If $\theta = 0$, then clearly the representation gives the identity. We can define the representation of the group in terms of the exponential using

$$D(\theta) = \mathrm{e}^{\mathrm{i}\theta X}.$$

The character of the group is defined in terms of the generators in the following sense. The generators satisfy a commutation relation we write as

$$[X_i, X_j] = \mathrm{i} f_{ijk} X_k.$$

This is called the Lie algebra of the group. The quantities f_{ijk} are called the structure constants of the group.

Example, SO(N). The group $\mathrm{SO}(N)$ are special orthogonal $N \times N$ matrices (satisfying $R^{\mathrm{T}}R = I$). The term special is a reference to the fact that these matrices have determinant $+1$. Let's turn to a familiar case, SO(3). This group has three parameters, the three angles defining rotations about the x, y, and z axes: $\theta_1, \theta_2, \theta_3$; then the following matrices are the representation of rotations in three dimensions

$$R(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix},\ R(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix},\ R(\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We get the generators

$$J_1 = -\mathrm{i}\frac{\partial R(\theta_1)}{\partial \theta_1}|_{\theta_1=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which we have the commutation relation for the Lie algebra with Levi-Civita symbol as the structure constants

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k.$$

This gives

$$(J_k)_{ij} = -\mathrm{i}\epsilon_{kij}.$$

3 Problems

Problem 1. Let $F_\mu = \partial_\mu \phi$ be a functional. Calculate the functional derivative $\frac{\delta F_\mu}{\delta \phi}$.

Proposed Solution

See Section 12.4.1 in Mandl & Shaw, especially Eq. (12.53)

$$\delta F[\phi(x)] = (F[\phi + \delta\phi] - F[\phi])\,(x) = \int \mathrm{d}^4 y \boxed{\frac{\delta F[\phi(x)]}{\delta \phi(x)}} \delta\phi(y). \qquad \text{MS (12.53)}$$

Observe

$$\begin{aligned} \frac{\delta F[\phi_1]}{\delta F[\phi_2]} &= \frac{\delta}{\delta F(\phi_2)} \int \mathrm{d}\phi \delta(\phi - \phi_1) F[\phi] && (3.1) \\ &= \delta(\phi_2 - \phi_1). && (3.2) \end{aligned}$$

and MS Eq. (12.55)

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^{(4)}(x - y). \qquad \text{MS (12.55)}$$

Using the definition, assuming the compact support for F_μ then integrating by parts

$$\begin{aligned} \delta F_\mu &= \partial_\mu \delta\phi = \frac{\partial}{\partial x^\mu}\delta\phi && (3.3) \\ &= \int \mathrm{d}^4 y \left(\frac{\partial}{\partial x^\mu}\delta\phi\right) \delta^{(4)}(y - x) && (3.4) \\ &= -\int \mathrm{d}^4 y \left(\frac{\partial}{\partial y^\mu}\delta^{(4)}(y - x)\right) \delta\phi && (3.5) \\ \text{cf. } \delta F[\phi(x)] &= \int \mathrm{d}y \boxed{\frac{\delta F[\phi(x)]}{\delta \phi(y)}} \delta\phi(y), && (3.6) \end{aligned}$$

thus we have

$$\frac{\delta F_\mu[\phi(x)]}{\delta \phi(y)} = -\frac{\partial}{\partial y^\mu}\delta^{(4)}(y - x). \qquad (3.7)$$

Problem 2. Find the Euler-Langrange EoM for the following Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi - \mathrm{i}eA_\mu \phi)\,(\partial^\mu \phi^* + \mathrm{i}eA^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \qquad (3.8)$$

Proposed Solution

This is the Lagrangian of a charged scalar field coupled to a photon field,

$$\mathcal{L} = D_\mu \phi D^\mu \phi - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $D_\mu \phi = \partial_\mu \phi - \mathrm{i}eA_\mu \phi$ (D_μ is the so-called covariant derivative obtained by minimal substitution).

We have the fields

$$\phi_a \in \{A_\mu, \phi, \phi^*\}. \qquad (3.9)$$

The Euler-Langrange equations of motion read

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)-\frac{\partial\mathcal{L}}{\partial\phi_a}=0, \tag{3.10}$$

with

$$\mathcal{L}=\underbrace{(\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)-m^2\phi^*\phi}_{\mathcal{L}^{\mathrm{SI}}}-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}. \tag{3.11}$$

We start with EoM for A_μ, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\mu A$. Here we know EoM for $-\frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}$ so we only do for $\mathcal{L}^{\mathrm{SI}}$,

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}^{\mathrm{SI}}}{\partial(\partial_\mu A_\nu)}\right) &= \partial_\mu\left(\frac{\partial}{\partial(\partial_\mu A_\mu)}\left((\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)\right)\right)=0, && (3.12)\\
\frac{\partial}{\partial A_\mu}\mathcal{L}^{\mathrm{SI}} &= \frac{\partial}{\partial A_\mu}\left((\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)\right) && (3.13)\\
&= \left(-\mathrm{i}e\delta^\mu_\rho\phi\right)\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)+(\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\mathrm{i}e\delta^\mu_\sigma\phi^*) && (3.14)\\
&= -\mathrm{i}e\phi\,(\partial_\mu\phi^*+\mathrm{i}eA_\mu\phi^*)+\mathrm{i}e\phi^*\,(\partial_\mu\phi-\mathrm{i}eA_\mu\phi)\,. && (3.15)
\end{aligned}$$

Hence, EoM for A_μ are

$$-\Box A_\mu+\partial^\rho\partial_\mu A_\rho+\mathrm{i}e\phi\,(\partial_\mu\phi^*+\mathrm{i}eA_\mu\phi^*)-\mathrm{i}e\phi^*\,(\partial_\mu\phi-\mathrm{i}eA_\mu\phi)=0. \tag{3.16}$$

For ϕ

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}^{\mathrm{SI}}}{\partial_\mu\phi}\right) &= \partial_\mu\left(\frac{\partial}{\partial_\mu\phi}\left((\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)-m^2\phi^*\phi\right)\right) && (3.17)\\
&= \partial_\mu\left(\delta^\mu_\rho\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)\right) && (3.18)\\
&= \partial_\mu\,(\partial^\mu\phi^*+\mathrm{i}eA^\mu\phi^*) && (3.19)\\
&= \partial_\mu\partial^\mu\phi^*+\mathrm{i}e\partial_\mu A^\mu\phi^*+\mathrm{i}eA^\mu\partial_\mu\phi^*, && (3.20)\\
\frac{\partial}{\partial\phi}\mathcal{L}^{\mathrm{SI}} &= \frac{\partial}{\partial\phi}\left((\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)-m^2\phi^*\phi\right) && (3.21)\\
&= (-\mathrm{i}eA_\rho)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)-m^2\phi^* && (3.22)\\
&= -\mathrm{i}eA^\rho\,(\partial_\rho\phi^*+\mathrm{i}eA_\rho\phi^*)-m^2\phi^* && (3.23)\\
&= -\mathrm{i}eA^\rho\partial_\rho\phi^*+e^2A^\rho A_\rho\phi^*-m^2\phi^*, && (3.24)
\end{aligned}$$

$$\begin{aligned}
\partial_\mu\partial^\mu\phi^*+\mathrm{i}e\partial_\mu A^\mu\phi^*+\mathrm{i}eA^\mu\partial_\mu\phi^*+\mathrm{i}eA^\rho\partial_\rho\phi^*-e^2A^\rho A_\rho\phi^*+m^2\phi^* &= 0 && (3.25)\\
\Box\phi^*+2\mathrm{i}e\partial_\mu A^\mu\phi^*+\mathrm{i}eA^\rho\partial_\rho\phi^*-e^2A^\rho A_\rho\phi^*+m^2\phi^* &= 0. && (3.26)
\end{aligned}$$

For ϕ^*

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}^{\mathrm{SI}}}{\partial_\mu\phi^*}\right) &= \partial_\mu\left(\frac{\partial}{\partial_\mu\phi^*}\left((\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\,(\partial_\sigma\phi^*+\mathrm{i}eA_\sigma\phi^*)-m^2\phi^*\phi\right)\right) && (3.27)\\
&= \partial_\mu\left((\partial_\rho\phi-\mathrm{i}eA_\rho\phi)\,\eta^{\rho\sigma}\delta^\mu_\sigma\right) && (3.28)\\
&= \partial_\mu\,(\partial^\mu\phi-\mathrm{i}eA^\mu\phi) && (3.29)\\
&= \partial_\mu\partial^\mu\phi-\mathrm{i}e\partial_\mu A^\mu\phi-\mathrm{i}eA^\mu\partial_\mu\phi, && (3.30)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial\phi}\mathcal{L}^{\mathrm{SI}} &= \frac{\partial}{\partial\phi}\Big(\left(\partial_\rho\phi - \mathrm{i}eA_\rho\phi\right)\eta^{\rho\sigma}\left(\partial_\sigma\phi^* + \mathrm{i}eA_\sigma\phi^*\right) - m^2\phi^*\phi\Big) && (3.31)\\
&= \left(\partial_\rho\phi - \mathrm{i}eA_\rho\phi\right)\eta^{\rho\sigma}\left(\mathrm{i}eA_\sigma\right) - m^2\phi && (3.32)\\
&= \mathrm{i}e\left(\partial_\rho\phi - \mathrm{i}eA_\rho\phi\right)A^\rho - m^2\phi && (3.33)\\
&= \mathrm{i}eA^\rho\partial_\rho\phi + e^2A_\rho A^\rho\phi - m^2\phi && (3.34)
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu\partial^\mu\phi - \mathrm{i}e\partial_\mu A^\mu\phi - \mathrm{i}eA^\mu\partial_\mu\phi - \mathrm{i}eA^\rho\partial_\rho\phi - e^2A_\rho A^\rho\phi + m^2\phi = 0 && (3.35)\\
\Box\phi - 2\mathrm{i}e\partial_\mu A^\mu\phi - \mathrm{i}eA^\mu\partial_\mu\phi - \mathrm{i}eA^\rho\partial_\rho\phi - e^2A_\rho A^\rho\phi + m^2\phi = 0. && (3.36)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
-\Box A_\mu + \partial^\rho\partial_\mu A_\rho + \mathrm{i}e\phi\left(\partial_\mu\phi^* + \mathrm{i}eA_\mu\phi^*\right) - \mathrm{i}e\phi^*\left(\partial_\mu\phi - \mathrm{i}eA_\mu\phi\right) &= 0, && (3.37)\\
\Box\phi^* + 2\mathrm{i}e\partial_\mu A^\mu\phi^* + \mathrm{i}eA^\rho\partial_\rho\phi^* - e^2A^\rho A_\rho\phi^* + m^2\phi^* &= 0, && (3.38)\\
\Box\phi - 2\mathrm{i}e\partial_\mu A^\mu\phi - \mathrm{i}eA^\mu\partial_\mu\phi - \mathrm{i}eA^\rho\partial_\rho\phi - e^2A_\rho A^\rho\phi + m^2\phi &= 0. && (3.39)
\end{aligned}
$$

Problem 3. Show that the Lagrangian density

$$
\mathcal{L} = \frac{1}{2}\left(\left(\partial\phi_1\right)^2 + \left(\partial\phi_2\right)^2\right) - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2 \qquad (3.40)
$$

is invariant under the transformation

$$
\begin{aligned}
\phi_1 &\to \phi_1' = \phi_1\cos\theta - \phi_2\sin\theta, && (3.41)\\
\phi_2 &\to \phi_2' = \phi_1\sin\theta + \phi_2\cos\theta. && (3.42)
\end{aligned}
$$

Find the corresponding Noether current and charge.

Proposed Solution

The simplest example of a filed theory exhibiting spontaneous symmetry breaking is the Goldstone model. Its Lagrangian density is given in Mandl and Shaw Eq. (18.3),

$$
\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi^*\right) - \frac{m^2}{2}\phi^*\phi - \frac{\lambda}{4}\left(\phi^*\phi\right)^2 \qquad (3.43)
$$

wherein $\phi(x)$ is a complex field. The Lagrangian we observe is obtained by substitution $\phi = \phi_1 + \mathrm{i}\phi_2$ and $\phi^* = \phi_1 - \mathrm{i}\phi_2$ where ϕ_1, ϕ_2 are two real fields.

This the internal transformation so we can only check the Lagrangian (internal transforamtions transform the fields into each other in some way without making reference to their dependence on space or time).

We begin with the kinetic term

$$
\begin{aligned}
\mathcal{L}^{\mathrm{K}} &= \frac{1}{2}\left(\left(\partial\phi_1\right)^2 + \left(\partial\phi_2\right)^2\right) && (3.44)\\
&= \frac{1}{2}\left(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2\right), && (3.45)\\
\to&\quad \text{(transforms to)}\\
\mathcal{L}'^{\mathrm{K}} &= \frac{1}{2}\left(\partial_\mu'\phi_1'\partial'^\mu\phi_1' + \partial_\mu'\phi_2'\partial'^\mu\phi_2'\right), && (3.46)\\
&\quad \text{(internal, so: } \partial_\mu' = \partial_\mu\text{)}
\end{aligned}
$$

$$
\begin{aligned}
\partial_\mu \phi'_1 &= \partial_\mu (\phi_1 \cos\theta - \phi_2 \sin\theta) && (3.47)\\
&= \partial_\mu \phi_1 \cos\theta - \partial_\mu \phi_2 \sin\theta, && (3.48)\\
\partial_\mu \phi'_2 &= \partial_\mu (\phi_1 \sin\theta + \phi_2 \cos\theta) && (3.49)\\
&= \partial_\mu \phi_1 \sin\theta + \partial_\mu \phi_2 \cos\theta && (3.50)
\end{aligned}
$$

Now substitute and evaluate

$$
\begin{aligned}
\mathcal{L}'^{\mathrm{K}} &= \frac{1}{2}(\partial_\mu\phi_1\cos\theta - \partial_\mu\phi_2\sin\theta)(\partial^\mu\phi_1\cos\theta - \partial^\mu\phi_2\sin\theta) + && (3.51)\\
&\quad + \frac{1}{2}(\partial_\mu\phi_1\sin\theta + \partial_\mu\phi_2\cos\theta)(\partial^\mu\phi_1\sin\theta + \partial^\mu\phi_2\cos\theta) && (3.52)\\
&= \frac{1}{2}[\partial_\mu\phi_1\cos\theta\partial^\mu\phi_1\cos\theta - \partial_\mu\phi_1\cos\theta\partial^\mu\phi_2\sin\theta - && (3.53)\\
&\quad -\partial_\mu\phi_2\sin\theta\partial^\mu\phi_1\cos\theta + \partial_\mu\phi_2\sin\theta\partial^\mu\phi_2\sin\theta + && (3.54)\\
&\quad +\partial_\mu\phi_1\sin\theta\partial^\mu\phi_1\sin\theta + \partial_\mu\phi_1\sin\theta\partial^\mu\phi_2\cos\theta + && (3.55)\\
&\quad +\partial_\mu\phi_2\cos\theta\partial^\mu\phi_1\sin\theta + \partial_\mu\phi_2\cos\theta\partial^\mu\phi_2\cos\theta] && (3.56)\\
&= \frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1\cos^2\theta + \partial_\mu\phi_2\partial^\mu\phi_2\sin^2\theta + && (3.57)\\
&\quad +\partial_\mu\phi_1\partial^\mu\phi_1\sin^2\theta + \partial_\mu\phi_2\partial^\mu\phi_2\cos^2\theta] && (3.58)\\
&= \frac{1}{2}[\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2] && (3.59)\\
&= \mathcal{L}^{\mathrm{K}}. && (3.60)
\end{aligned}
$$

Note that $\phi_1^2 + \phi_2^2$ shares the same pattern! Thus we have

$$
\phi_1'^2 + \phi_2'^2 = \phi_1^2 + \phi_2^2 \qquad (3.61)
$$

Hence the Lagrangian does not change

$$
\mathcal{L}' = \mathcal{L}.
$$

Now, the infinitesimal variations of the fields ϕ_a, $a = 1, 2$, are $\delta_\epsilon\phi_a(x) = \phi'_a(x') - \phi_a(x)$,

$$
\begin{aligned}
\delta\phi_1 &= -\theta\phi_2, \quad (\phi_1\cos\theta - \phi_2\sin\theta \approx -\theta\phi_2 \text{ for small } \theta) && (3.62)\\
\delta\phi_2 &= \theta\phi_1. \quad (\phi_1\cos\theta - \phi_2\sin\theta \approx -\theta\phi_2 \text{ for small } \theta) && (3.63)
\end{aligned}
$$

Substituting into the expression for the Noether current we have, for $\epsilon \in \{\theta\}$,

$$
\begin{aligned}
J^\mu_\epsilon &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta_\epsilon\phi_a - T^\mu{}_\nu \underbrace{\delta_\epsilon x^\nu}_{0 \text{ for internal}} && (3.64)\\
J^\mu_\theta &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\delta\phi_2. && (3.65)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\left\{\frac{1}{2}(\partial_\rho\phi_1\partial^\rho\phi_1 + \partial_\rho\phi_2\partial^\rho\phi_2)\right\} = \partial^\mu\phi_1, && (3.66)\\
\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu 2_1)}\left\{\frac{1}{2}(\partial_\rho\phi_1\partial^\rho\phi_1 + \partial_\rho\phi_2\partial^\rho\phi_2)\right\} = \partial^\mu\phi_2, && (3.67)
\end{aligned}
$$

$$
J^\mu_\theta = -\theta\partial^\mu\phi_1\phi_2 + \theta\phi_1\partial^\mu\phi_2, \qquad (3.68)
$$

$$J^\mu = -\partial^\mu\phi_1\phi_2 + \phi_1\partial^\mu\phi_2. \tag{3.69}$$

The Noether curren tis

$$Q_\epsilon = \int \mathrm{d}^3x\, J^0_\epsilon = \int \mathrm{d}^3x \left[-\theta\partial^0\phi_1\phi_2 + \theta\phi_1\partial^0\phi_2\right] \tag{3.70}$$

$$= \theta\int \mathrm{d}^3x \left[\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2\right]. \tag{3.71}$$

The charge corresponds to SO(2) symmetry, which is isomorphic to U(1) that is the global symmetry of a complex scalar field (the equivalent form that we mentioned in the introduction).

Problem 4. The Lagrangian density of a real three-component scalar field is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\boldsymbol{\phi}^{\mathrm{T}}(x)\right)\left(\partial^\mu\boldsymbol{\phi}(x)\right) - \frac{m^2}{2}\boldsymbol{\phi}^{\mathrm{T}}(x)\boldsymbol{\phi}(x) \quad \text{where } \boldsymbol{\phi}(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\\ \phi_3(x)\end{pmatrix}. \tag{3.72}$$

Find the equations of motions for the scalar fields $\phi_a(x)$. Prove that the Lagrangian density is SO(3) invariant and find the Noether currents.
Proposed Solution

We expand the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\sum_a \left(\partial_\mu\phi_a\right)\left(\partial^\mu\phi_a\right) - \frac{m^2}{2}\sum_a \phi_a^2. \tag{3.73}$$

It is obvious that

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) = \partial_\mu\frac{\partial}{\partial(\partial_\mu\phi_a)}\left(\frac{1}{2}\sum_b \left(\partial_\rho\phi_b\right)\left(\partial^\rho\phi_b\right) - \frac{m^2}{2}\sum_b \phi_b^2\right) \tag{3.74}$$

$$= \partial_\mu\partial^\mu\phi_a, \tag{3.75}$$

$$\frac{\partial\mathcal{L}}{\partial\phi_a} = \frac{\partial}{\partial\phi_a}\left(\frac{1}{2}\sum_b \left(\partial_\rho\phi_b\right)\left(\partial^\rho\phi_b\right) - \frac{m^2}{2}\sum_b \phi_b^2\right) \tag{3.76}$$

$$= -m^2\phi_a, \tag{3.77}$$

and the equations of motion are

$$\partial_\mu\partial^\mu\phi_a + m^2\phi_a = 0, \quad a \in \{1,2,3\}. \tag{3.78}$$

The expression $\sum_b \phi_b^2$ is invariant under SO(3) transformations (it shares the behavor of the 3-d vectors with the components ϕ_b), hence the Langrangian density has the same symmetry.

The generators of SO(3) group are (recall the recap)

$$J_1 = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\mathrm{i}\\ 0 & \mathrm{i} & 0\end{pmatrix}, \quad J_2 = \begin{pmatrix}0 & 0 & -\mathrm{i}\\ 0 & 0 & 0\\ \mathrm{i} & 0 & 0\end{pmatrix}, \quad J_3 = \begin{pmatrix}0 & -\mathrm{i} & 0\\ \mathrm{i} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.79}$$

can be written

$$(J_a)_{ij} = -\mathrm{i}\epsilon_{aij}. \tag{3.80}$$

Under SO(3) transformations, the infinitesimal variations of the fields are (note the sums over repeated indices)

$$\begin{aligned} \phi_i' = \phi_i + \delta\phi_i &= \phi_i + \mathrm{i}(J_a)_{ij}\theta_a\phi_j = \phi_i + \epsilon_{aij}\theta_a\phi_j & (3.81)\\ \delta\phi_i &= \epsilon_{aij}\theta_a\phi_j. & (3.82)\end{aligned}$$

The Noether current is

$$\begin{aligned} J^\mu_\epsilon &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta_\epsilon\phi_a - T^\mu{}_\nu \underbrace{\delta_\epsilon x^\nu}_{\text{0 for internal}} & (3.83)\\ J^\mu_\theta &= \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i & (3.84)\\ &= \partial^\mu\phi_i\,\epsilon_{aij}\theta_a\phi_j & (3.85)\\ &= \theta_a\,\epsilon_{aij}\phi_j\partial^\mu\phi_i & (3.86)\\ &= -\theta_a\,\epsilon_{aij}\phi_i\partial^\mu\phi_j & (3.87)\\ &= -\boldsymbol{\theta}\cdot(\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}). & (3.88)\end{aligned}$$

The prameters of rotations θ_k are arbitrary and therefore we have the currents

$$\mathbf{J}^\mu = -\boldsymbol{\phi}\times\partial^\mu\boldsymbol{\phi}. \quad (3.89)$$

Problem 5. Consider the Lagrangian density of a real three-component scalar field is given by

$$\mathcal{L} = (\partial_\mu\boldsymbol{\phi}^*(x))\,(\partial^\mu\boldsymbol{\phi}(x)) - m^2\boldsymbol{\phi}^{\mathrm{T}}(x)\boldsymbol{\phi}(x) \quad \text{where } \boldsymbol{\phi}(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix}. \quad (3.90)$$

Show that the Lagrangian density has SU(2) symmetry. Find the related Noether currents and charges.

Proposed Solution

Under the SU(2) transformations, the fields are transformed according

$$\phi' = \exp\left(\frac{\mathrm{i}}{2}\tau^a\theta_a\right)\phi, \quad (3.91)$$

where τ_a $(a = 1,2,3)$ are the Pauli matrices. For an infinitesimal transformation we obtain

$$\delta\phi_i = \frac{\mathrm{i}}{2}\tau^a_{ij}\theta_a\phi_j, \qquad \delta\phi_i^* = -\frac{\mathrm{i}}{2}\tau^a_{ij}\theta_a\phi_j^*. \quad (3.92)$$

The Noether current is determined by (here $\epsilon = \{\theta^a\}$)

$$\begin{aligned} J^\mu_\epsilon &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta_\epsilon\phi_a - T^\mu{}_\nu \underbrace{\delta_\epsilon x^\nu}_{\text{0 for internal}} & (3.93)\\ J^\mu_{\{\theta^a\}} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i^*)}\delta\phi_i^* & (3.94)\\ &= \sum_a \frac{\mathrm{i}}{2}\theta^a\left(\partial^\mu\phi_i^*\tau^a_{ij}\phi_j - \phi_i^*\tau^a_{ij}\partial^\mu\phi_j\right) & (3.95)\\ &= \sum_a \theta^a J^\mu_a. & (3.96)\end{aligned}$$

From the previous relation (θ^a are independent parameters spanning a vector space) it follows that the conserved currents are (up to a sign)

$$J_a^\mu = \frac{\mathrm{i}}{2}\left(\partial^\mu \phi_i^* \tau_{ij}^a \phi_j - \phi_i^* \tau_{ij}^a \partial_\mu \phi_j\right). \tag{3.97}$$

The charges are

$$Q_a = \frac{\mathrm{i}}{2}\int \mathrm{d}^3x \left(\partial^0 \phi_i^* \tau_{ij}^a \phi_j - \phi_i^* \tau_{ij}^a \partial^0 \phi_j\right)$$