Tutorial 7 - The Dirac field (continued)

Olga Bessidskaia Bylund

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1 The Dirac field

1.1 Introduction

The second part of the problems for the Dirac field. A couple of the questions are phrased in a way to act as support during the tutorial and may be difficult to understand when just reading them.

In Mandl and Shaw, this material corresponds to chapter 4 and part of the appendix. Additional reading for deeper understanding is suggested at the end of the document.

1.2 Main background

The Lagrangian density of the Dirac field is:

$$L = c\bar{\psi}(x)\left[i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - mc\right]\psi(x)\ . \tag{1}$$

This Lagrangian describes fermions. ψ has 4 components. γ^{μ} is a set of four 4x4 matrices. We often use natural units, where $c = 1$ and $\hbar = 1$.

The field $\psi(x)$ can be decomposed as:

$$\psi(x) = \sum_{r,\mathbf{p}}[c_r(\mathbf{p})u_r(\mathbf{p})\mathrm{e}^{-ipx/\hbar} + d_r^{\dagger}(\mathbf{p})v_r(\mathbf{p})\mathrm{e}^{ipx/\hbar}] \tag{2}$$

and similarly

$$\bar{\psi}(x) = \sum_{r,\mathbf{p}}[d_r(\mathbf{p})\bar{v}_r(\mathbf{p})\mathrm{e}^{-ipx/\hbar} + c_r^{\dagger}(\mathbf{p})\bar{u}_r(\mathbf{p})\mathrm{e}^{ipx/\hbar}]\,, \tag{3}$$

with $\bar{u}_r(\mathbf{p}) = u_r(\mathbf{p})^{\dagger}\gamma^0$ etc.

The annihilation and creation operators in these expressions have the following *anti*-commutation properties:

$$[c_r, c_s^{\dagger}]_+ = \delta_{rs}, \quad [c_r, c_s]_+ = [c_r^{\dagger}, c_s^{\dagger}]_+ = 0\,, \tag{4}$$

$$[d_r, d_s^\dagger]_+ = \delta_{rs}, \quad [d_r, d_s]_+ = [d_r^\dagger, d_s^\dagger]_+ = 0\,. \tag{5}$$

All anti-commutators between c-operators and d-operators vanish.

The Dirac equation is:

$$\left[i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right]\psi(x) = 0\,. \tag{6}$$

1.3 Lorentz tranformations

The Lorentz transformation Λ^μ_ν acts on spacetime x^ν and hence the inverse acts on the four-momentum $p_\mu = i\frac{\partial}{\partial x^\mu}$. With Peskin and Shroeder notation (**chiral representation**), $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]_+ = \sigma^{\mu\nu}/2$ contains the generators for boosts (S^{0i}) and for rotations (S^{ij}). A four component field ψ that transforms under boots and rotations like this (in this particular representation) is a Dirac spinor. $S^{\mu\nu}$ is a representation of the Lorentz algebra. It is not unitary.

The Lorentz transformation Λ^μ_ν acts on spacetime. The spinor representation of the Lorentz transformation is

$$S = \exp(-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu})\,. \tag{7}$$

S acts on Dirac spinors S and Λ and S commute with each other.

So, under Lorentz transformations, we have:

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu\,, \tag{8}$$

$$\psi(x) \to \psi'(x') = S\psi(x)\,, \tag{9}$$

$$\partial_\mu \to \partial'_\mu = (\Lambda^{-1})^\nu_\mu \partial_\nu \tag{10}$$

These are eqs A.48 (+ inverted) and A.51 respectively in Mandl and Shaw.

The matrix S is constructed to have the following properties:

$$\gamma^\nu = \Lambda^\nu_\mu S\gamma^\mu S^{-1}\,, \quad S\gamma^\lambda S^{-1} = \gamma^\nu \Lambda^\Lambda_\nu \tag{11}$$

$$S^{-1} = \gamma^0 S^\dagger \gamma^0\,. \tag{12}$$

These are equations A.49, A.56 and A.50 in Mandl and Shaw. In P&S, Eq. (11) is described as "the γ matrices are invariant under simultaneous rotations of their vector indices μ and spinor indices".

As will be shown in the first exercise, under a Lorentz transformation Λ, given the identity for γ Eq. (11), ψ should transform as $\psi \to S\psi$ to give a Lorentz invariant expression.

1.4 The exercises

1. Demonstrate the Lorentz invariance of the Dirac equation.
2. Check whether $\psi^\dagger(x)\psi(x)$ and $\bar{\psi}(x)\psi(x)$ are Lorentz invariant.
3. Solve the Dirac equation for a free particle.
4. Verify the properties of the energy projection operator:

$$\Lambda_{+-}(\mathrm{p}) = \frac{\pm \not{p} + m}{2m} \tag{13}$$

   by showing that $\Lambda_\pm^2 = \Lambda_\pm$ and $\Lambda_+\Lambda_- = 0$. How do these projectors act on the sipnors $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$?

5. Show that $\sum u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{\not{p}+m}{2m}$ and $\sum v_r(\mathbf{p})\bar{v}_r(\mathbf{p}) = \frac{\not{p}-m}{2m}$.
6. Prove that if $\psi(x)$ satisfies the Dirac equation, it is also a solution of the Klein-Gordon equation.
7. Reflect over how the property $v^\dagger(-\mathbf{p})u_r(\mathbf{p}) = 0$ can be used in expressions similar to the one above.

1.5 Further reading

- Peskin and Shroeder, "An introduction to QFT", chapter 3 - The Dirac equation and Lorentz invariance (borrowed heavily from there).
- Gross, "Relativistic QM and QFT", chapter 5.1 - how the γ^μ matrices are constructed (extra).
- Schwartz, "QFT and the SM", chapter 10 and 11 on spinors and chapter 12 on the spin statistics theorem (extra).