Tutorial 3 – Solutions

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Topics:

- Appendix: more on index gymnastics in SR
- Fourier transform in QM and QFT.
- EoM for complex and real KGF.
- Quantum harmonic oscillator (QHO), ladder operators and Heisenberg Algebra

1 Appendix SR: Recap III

The components of vectors are denoted by indices up (for example, A^μ) while the components of dual vectors (covectors) are denoted by indices down (for example, p_μ). Typical vectors are position,

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, x^i) = (t, \mathbf{x}), \tag{1.1}$$

and momentum $p = \hbar k$,

$$p^\mu = (p^0, p^1, p^2, p^3) = (p^0, p^i) = (E, \mathbf{p}), \tag{1.2}$$
$$k^\mu = (k^0, k^1, k^2, k^3) = (k^0, k^i) = (\omega, \mathbf{k}). \tag{1.3}$$

A typical covector is a gradient of a scalar field. For example, for a scalar field $f(x)$,

$$\partial_\mu f(x) = \frac{\partial f}{\partial x^\mu} = (\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f) = (\partial_0 f, \partial_i f), \tag{1.4}$$

The corresponding total differential is,

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu = \partial_\mu f\,\mathrm{d}x^\mu. \tag{1.5}$$

Remark 1. If the units of vectors are L, then the units of covectors are L^{-1} (this is clear from x^μ and $\partial/\partial x^\mu$).

Remark 2. The four-momenta naturally lives in the dual space. Their nature is more obvious in QM identifying the momentum operator as $\mathrm{i}p_\mu = \partial_\mu$.

The arena of SR is endowed with the Minkowski metric η,

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.6}$$

The metric is used to measure lengths of vectors. For example,

$$k^2 = \eta_{\mu\nu}k^\mu k^\nu = (k^0)^2 - \sum_{i=1}^{3}(k^i)^2 = \omega^2 - \mathbf{k}\cdot\mathbf{k} = \omega^2 - |\mathbf{k}|^2\,. \tag{1.7}$$

The metric is also used to transform vectors into covectors ("lower indices"). For example,

$$x_\mu = \eta_{\mu\nu} x^\nu = (t, -\mathbf{x}), \tag{1.8}$$

$$p_\mu = \eta_{\mu\nu} p^\nu = (E, -\mathbf{p}) \tag{1.9}$$

$$k_\mu = \eta_{\mu\nu} k^\nu = (\omega, -\mathbf{k}). \tag{1.10}$$

Then we can write,

$$k^2 = \eta_{\mu\nu} k^\mu k^\nu = k_\nu k^\nu. \tag{1.11}$$

Note that metric is always symmetric in indices.

The inverse of the metric is $(\eta^{-1})^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu_\nu$ for which we use the same symbol $\eta^{\mu\nu} := (\eta^{-1})^{\mu\nu}$. The inverse of the metric has the same components,

$$\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1). \tag{1.12}$$

The inverse metric is used to transform covectors into vectors ("raise indices"). For example,

$$\partial^\mu f = \eta^{\mu\nu}\partial_\nu f = (\frac{\partial f}{\partial t}, -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial z}) = (\partial_t f, -\boldsymbol{\nabla} f) = (\partial_t f, -\partial_i f). \tag{1.13}$$

Note that here we wrote $\boldsymbol{\nabla} f$ as $\partial_i f$ making a bit of confusion since $\boldsymbol{\nabla}$ conventionally denotes a spatial *vector* while $\partial_i f$ denotes the spatial part of a *covector*.

Coordinate transformations

Consider a coordinate transformation $x^\mu \to y^\mu = y^\mu(x)$. Then,

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\mathrm{d}x^\mu \tag{1.14}$$

$$= \frac{\partial f}{\partial x^\mu}\delta^\mu_\nu \mathrm{d}x^\nu \qquad \left(\delta^\mu_\nu = \frac{\partial x^\mu}{\partial y^\rho}\frac{\partial y^\rho}{\partial x^\nu}\right) \tag{1.15}$$

$$= \frac{\partial f}{\partial x^\mu}\left(\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial y^\rho}{\partial x^\nu}\right)\mathrm{d}x^\nu \tag{1.16}$$

$$= \left(\frac{\partial f}{\partial x^\mu}\frac{\partial x^\mu}{\partial y^\rho}\right)\left(\frac{\partial y^\rho}{\partial x^\nu}\mathrm{d}x^\nu\right) \tag{1.17}$$

$$= \frac{\partial f}{\partial y^\mu}\mathrm{d}y^\nu. \tag{1.18}$$

The Jacobian A and its inverse $B = A^{-1}$ are,

$$A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu}, \qquad B^\mu{}_\nu = \frac{\partial y^\mu}{\partial x^\nu}. \tag{1.19}$$

To see how this works, let us consider a concrete example in two dimensions (so the indices runs $\mu, \nu = 0, 1$) with the coordinate transformations,

$$\begin{cases} y^0 &= x^0 + x^1, \\ y^1 &= x^0 - x^1, \end{cases} \qquad \begin{cases} x^0 &= \frac{1}{2}\left(y^0 + y^1\right), \\ x^1 &= \frac{1}{2}\left(y^0 - y^1\right). \end{cases} \tag{1.20}$$

In matrix notation we have,

$$A^\mu{}_\nu = \begin{pmatrix} A^0{}_0 & A^0{}_1 \\ A^1{}_0 & A^1{}_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x^0}{\partial y^0} & \frac{\partial x^0}{\partial y^1} \\ \frac{\partial x^1}{\partial y^0} & \frac{\partial x^1}{\partial y^1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{1.21}$$

$$B^\mu{}_\nu = \begin{pmatrix} B^0{}_0 & B^0{}_1 \\ B^1{}_0 & B^1{}_1 \end{pmatrix} = \begin{pmatrix} \frac{\partial y^0}{\partial x^0} & \frac{\partial y^0}{\partial x^1} \\ \frac{\partial y^1}{\partial x^0} & \frac{\partial y^1}{\partial x^1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{1.22}$$

Here we used to denote column vectors by up-indices and row vectors by down-indices. Observe that $\det A = -2$ and $\det B = -1/2$.

Let us calculate the components of $\frac{\partial x^\mu}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\nu}$, i.e.,

$$C^\mu{}_\nu = A^\mu{}_\rho B^\rho{}_\nu = \sum_\rho A^\mu{}_\rho B^\rho{}_\nu = A^\mu{}_0 B^0{}_\nu + A^\mu{}_1 B^1{}_\nu. \tag{1.23}$$

Explicit expansion of the sums falls back to multiplication of matrices,

$$C^\mu{}_\nu = \begin{pmatrix} A^0{}_0 & A^0{}_1 \\ A^1{}_0 & A^1{}_1 \end{pmatrix} \cdot \begin{pmatrix} B^0{}_0 & B^0{}_1 \\ B^1{}_0 & B^1{}_1 \end{pmatrix} = \begin{pmatrix} A^0{}_0 B^0{}_0 + A^0{}_1 B^1{}_0 & A^0{}_0 B^0{}_1 + A^0{}_1 B^1{}_1 \\ A^1{}_0 B^0{}_0 + A^1{}_1 B^1{}_0 & A^1{}_0 B^0{}_1 + A^1{}_1 B^1{}_1 \end{pmatrix} \tag{1.24}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} - \left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.25}$$

That is,

$$\delta^\mu_\nu = \frac{\partial x^\mu}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\nu}. \tag{1.26}$$

Caution. If you use matrix notation, then be sure to match the row/columns of the matrices that will be summed across when multiplied. This sometimes requires usage of transposition. For example, in 4D,

$$k^\mu = \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix}, \qquad \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.27}$$

yields $k^2 = \eta_{\mu\nu}k^\mu k^\nu = k^\mu \eta_{\mu\nu} k^\nu$ in matrix notation $k^2 = k^\mathrm{T}\eta k$,

$$k^2 = \begin{pmatrix} k^0 & k^1 & k^2 & k^3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix} \tag{1.28}$$

$$= \begin{pmatrix} k^0 & k^1 & k^2 & k^3 \end{pmatrix} \cdot \begin{pmatrix} k^0 \\ -k^1 \\ -k^2 \\ -k^3 \end{pmatrix} = k^0k^0 - k^1k^1 - k^2k^2 - k^3k^3. \tag{1.29}$$

2 Fourier transform

The Fourier transform has many forms, where in physics, we mostly use the one written in terms of angular frequency ω,

$$\hat{f}(\omega) = \int_{\mathbb{R}^n} \mathrm{d}^n x \, f(x) \, \mathrm{e}^{-\mathrm{i}\omega\cdot x}. \tag{2.1}$$

Under this convention, the inverse transform becomes,

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{d}^n \omega \, \tilde{f}(\omega) \mathrm{e}^{\mathrm{i}\omega\cdot x}. \tag{2.2}$$

When the Fourier transform is defined this way, it is no longer a unitary transformation on $L^2(\mathbb{R}^n)$. There is also less symmetry between the formulae for the Fourier transform and its inverse. Another convention is to split the factor of $(2\pi)^n$ evenly between the Fourier transform and its inverse, which leads to definitions (convention used in Sakurai, e.g.),

$$\tilde{f}(\omega) = \frac{1}{(2\pi)^{n/2}} \int \mathrm{d}^n x \, f(x) \, \mathrm{e}^{-\mathrm{i}\omega\cdot x}, \quad f(x) = \frac{1}{(2\pi)^{n/2}} \int \mathrm{d}^n \omega \, \tilde{f}(\omega) \mathrm{e}^{\mathrm{i}\omega\cdot x}. \tag{2.3}$$

Fourier transform in QM (using braket notation)

In QM, we can consider a wave function in the position space (the configuration space) and in the momentum space.

In a one-dimensional configuration space we denote,

$$\langle x|\psi\rangle = \psi(x). \tag{2.4}$$

The orthogonality and completeness conditions for the configuration space reads,

$$\langle x'|x''\rangle = \delta(x' - x''), \tag{2.5}$$

$$\int \mathrm{d}x \, |x\rangle \langle x| = \mathbb{1}. \tag{2.6}$$

In a corresponding momentum space we denote,

$$\langle p|\psi\rangle = \tilde{\psi}(p). \tag{2.7}$$

The orthogonality and completeness conditions for momentum space reads,

$$\langle p'|p''\rangle = \delta(p' - p''), \tag{2.8}$$

$$\int \mathrm{d}p \, |p\rangle \langle p| = \mathbb{1}. \tag{2.9}$$

We use the Fourier transform to go between different bases, using the transformation function (in natural units, $c = \hbar = 1$),

$$\langle x|p\rangle = \frac{\exp(\mathrm{i}px)}{\sqrt{2\pi}}, \tag{2.10}$$

$$\langle p|x\rangle = \langle x|p\rangle^* = \frac{\exp(-\mathrm{i}px)}{\sqrt{2\pi}}. \tag{2.11}$$

This normalization of $\langle x|p\rangle$ is symmetric so that,

$$\langle x'|x''\rangle = \langle x'| \boxed{\int \mathrm{d}p \, |p\rangle \langle p|} \, |x''\rangle \tag{2.12}$$

$$= \int \mathrm{d}p \, \langle x' \, |p\rangle \langle p| \, x''\rangle \tag{2.13}$$

$$= \int \mathrm{d}p \, \frac{\exp(\mathrm{i}px')}{\sqrt{2\pi}} \frac{\exp(-\mathrm{i}px'')}{\sqrt{2\pi}} \tag{2.14}$$

$$= \int \frac{\mathrm{d}p}{2\pi} \exp[\mathrm{i}p(x' - x'')] = \delta(x' - x''). \tag{2.15}$$

Note that, in the discrete case, the integrals become sums and Dirac delta functions become Kronecker deltas,

$$\sum_n |n\rangle \langle n| = \mathbb{1}, \quad \langle n|m\rangle = \delta_{nm}, \quad \langle n|\psi\rangle = \psi_n. \tag{2.16}$$

In three dimensions, the above expressions generalizes to,

$$\langle \mathbf{x}|\psi\rangle = \psi(\mathbf{x}), \tag{2.17}$$

$$\langle \mathbf{x}'|\mathbf{x}''\rangle = \delta^3(\mathbf{x}' - \mathbf{x}''), \tag{2.18}$$

$$\int \mathrm{d}^3\mathbf{x} \, |\mathbf{x}\rangle \langle \mathbf{x}| = \mathbb{1}, \tag{2.19}$$

$$\langle \mathbf{p}|\psi\rangle = \tilde{\psi}(\mathbf{p}), \tag{2.20}$$

$$\langle \mathbf{p}'|\mathbf{p}''\rangle = \delta^3(\mathbf{p}' - \mathbf{p}''), \tag{2.21}$$

$$\int \mathrm{d}^3\mathbf{p} \, |\mathbf{p}\rangle \langle \mathbf{p}| = \mathbb{1}, \tag{2.22}$$

$$\langle \mathbf{x}|\mathbf{p}\rangle = \frac{1}{(2\pi)^{3/2}} \cdot \exp(\mathrm{i}\mathbf{p} \cdot \mathbf{x}), \tag{2.23}$$

$$\langle \mathbf{p}|\mathbf{x}\rangle = \langle \mathbf{x}|\mathbf{p}\rangle^* = \frac{1}{(2\pi)^{3/2}} \cdot \exp\left(-\mathrm{i}\mathbf{p}\cdot\mathbf{x}\right). \tag{2.24}$$

Fourier transform in QFT

In the four-dimensional configuration space x^μ, we have,

$$\phi(x) = \int \mathrm{d}^4k\, \tilde{\phi}(k)\, \mathrm{e}^{\mathrm{i}kx}, \qquad \text{where } kx \equiv k_\mu x^\mu. \tag{2.25}$$

Note that, after quantization, the scalar field $\phi(x)$ becomes an *operator* in QFT (while in QM $\psi(x)$ was a wave function).

The inverse Fourier transform is,

$$\tilde{\phi}(k) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4x\, \phi(x)\, \mathrm{e}^{-\mathrm{i}kx}. \tag{2.26}$$

The normalization constant $\frac{1}{(2\pi)^4}$ (or $\frac{1}{(2\pi)^3}$ in three dimensions) can be split across the above two expressions. For example, compare:

- Mandl and Shaw, Eq. (3.7)
  (N.b. $\phi(x)$ is an operator , as well as a, $a^\dagger$!)

$$\phi(x) = \phi^+(x) + \phi^-(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} \left(a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right) \tag{2.27}$$

$$\phi^+(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a(\mathbf{k})\, \mathrm{e}^{-\mathrm{i}kx} \tag{2.28}$$

$$\phi^+(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a^\dagger(\mathbf{k})\, \mathrm{e}^{\mathrm{i}kx} \tag{2.29}$$

- Schwartz, Eq. (2.78)

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}px} + a_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}px}\right) \tag{2.30}$$

- Peskin and Schroeder Eq. (2.47)

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}}\mathrm{e}^{-\mathrm{i}p\cdot x} + a_{\mathbf{p}}^\dagger \mathrm{e}^{\mathrm{i}p\cdot x}\right)|_{p^0=E_{\mathbf{p}}} \tag{2.31}$$

The above are the plane wave solutions $\phi(x)$ for the real Klein-Gordon-Fock equation,

$$(\Box + m^2)\phi(x) = (\partial_\mu\partial^\mu + m^2)\phi(x) = 0, \quad \partial_\mu\partial^\mu = \partial_t^2 - \boldsymbol{\nabla}^2, \tag{2.32}$$

which is derived as the equations of motion from the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi(x)\right)\left(\partial^\mu\phi(x)\right) - \frac{1}{2}m^2\phi^2(x). \tag{2.33}$$

3 Equations of motion for the real and complex KGF

For the Lagrangian density $\mathcal{L}$ which depends on a set of scalar fields $\phi_a(x)$ and their first derivatives, the equations of motion follow from the Euler-Lagrange equation,

$$\frac{\partial}{\partial \phi_a}\mathcal{L} - \partial_\mu \frac{\partial}{\partial(\partial_\mu \phi_a)}\mathcal{L} = 0. \tag{3.1}$$

For the Lagrangian density (2.33) of the real KGF field, we have,

$$\frac{\partial}{\partial \phi}\left\{\frac{1}{2}\partial_\rho \phi \partial^\rho \phi - \frac{1}{2}m^2\phi^2\right\} = -m^2\phi, \tag{3.2}$$

$$\frac{\partial}{\partial(\partial_\mu \phi)}\left\{\frac{1}{2}\partial_\rho \phi \partial^\rho \phi - \frac{1}{2}m^2\phi^2\right\} = \partial^\mu \phi. \tag{3.3}$$

Hence, (3.1) yields,

$$-(\partial_\mu \partial^\mu + m^2)\phi(x) = 0. \tag{3.4}$$

For the complex KGF field we have the Lagrangian density,

$$\mathcal{L} = (\partial_\mu \phi^*(x))(\partial^\mu \phi(x)) - m^2\phi^*(x)\phi(x). \tag{3.5}$$

Here the two fields $\phi(x)$ and $\phi^*(x)$ (the complex conjugate of ϕ) are considered to be independent. The equivalent way of writing the complex KGF is expressing the complex field $\phi(x)$ in terms of two real fields $\phi_1(x)$ and $\phi_2(x)$,

$$\phi(x) = \phi_1(x) + \mathrm{i}\phi_2(x), \tag{3.6}$$

$$\phi^*(x) = \phi_1(x) - \mathrm{i}\phi_2(x), \tag{3.7}$$

in which case the Lagrangian density of the complex field can be written as the sum of two Lagrangian densities of the real fields,

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu \phi_1(x))(\partial^\mu \phi_1(x)) - \frac{1}{2}m^2\phi_1^2(x)\\ &+ \frac{1}{2}(\partial_\mu \phi_1(x))(\partial^\mu \phi_2(x)) - \frac{1}{2}m^2\phi_2^2(x).\end{aligned} \tag{3.8}$$

The EoM of the complex KGF can be calculated using,

$$\frac{\partial}{\partial \phi}\left\{\partial_\rho \phi^* \partial^\rho \phi - m^2\phi^*\phi\right\} = -m^2\phi^*, \tag{3.9}$$

$$\frac{\partial}{\partial \phi^*}\left\{\partial_\rho \phi^* \partial^\rho \phi - m^2\phi^*\phi\right\} = -m^2\phi, \tag{3.10}$$

$$\frac{\partial}{\partial(\partial_\mu \phi)}\left\{\partial_\rho \phi^* \partial^\rho \phi - m^2\phi^*\phi\right\} = \partial^\mu \phi^*, \tag{3.11}$$

$$\frac{\partial}{\partial(\partial_\mu \phi^*)}\left\{\partial_\rho \phi^* \partial^\rho \phi - m^2\phi^*\phi\right\} = \partial^\mu \phi. \tag{3.12}$$

This gives the complex KGF equation,

$$\begin{cases} \left[\partial_\mu \partial^\mu + m^2\right] \phi(x) & = 0, \\ \left[\partial_\mu \partial^\mu + m^2\right] \phi^*(x) & = 0. \end{cases} \tag{3.13}$$

Note on the asymmetric form of the Lagrangian density

The equations of motion are not changed if one adds a total derivative to the Lagrangian density that vanishes on the boundary after the variation. This can be used to write the Lagrangian density in terms of EoM.

Namely, consider rewriting the Lagrangian density of the real KGF field using,

$$\partial_\mu \left(\phi \partial^\mu \phi\right) = \left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) + \phi\, \partial_\mu \left(\partial^\mu \phi\right), \tag{3.14}$$

that is, using $\partial_\mu \phi \partial^\mu \phi = \partial_\mu \left(\partial^\mu \phi\right) - \phi\, \Box \phi$ so that,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left[\partial_\mu \left(\phi \partial^\mu \phi\right) - \phi \Box \phi\right] - \frac{1}{2} m^2 \phi^2 \\ &= \frac{1}{2}\partial_\mu \left(\phi \partial^\mu \phi\right) - \frac{1}{2}\left[\phi\, \Box \phi + m^2 \phi^2\right] \\ &= \frac{1}{2}\partial_\mu \left(\phi \partial^\mu \phi\right) - \frac{1}{2}\phi \left[\Box + m^2\right] \phi. \end{aligned}$$

The Lagrangian density of the real KGF field is then equal to,

$$\mathcal{L}' = -\frac{1}{2}\phi \left[\Box + m^2\right] \phi, \tag{3.15}$$

modulo a total derivative. Observe that this Lagrangian is in asymmetric form with respect to the field (similarly to the one for the Dirac field). The variation of the such a Lagrangian density is simple and yields directly EoM.

Applying the same procedure on the complex KGF field, we have,

$$\mathcal{L}' = -\frac{1}{2}\left\{\phi^* \left[\Box + m^2\right] \phi + \phi \left[\Box + m^2\right] \phi^*\right\}. \tag{3.16}$$

Fourier transform of the real KGF

Now back to the real KGF equation. It is easy to see that for (2.25),

$$(\Box + m^2)\phi(x) = (\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2) \int \mathrm{d}^4 k\, \tilde{\phi}(k)\, \mathrm{e}^{\mathrm{i}kx} \tag{3.17}$$

$$= \int \mathrm{d}^4 k\, \tilde{\phi}(k)\, \left\{(\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2)\mathrm{e}^{\mathrm{i}kx}\right\}, \tag{3.18}$$

we have,

$$\partial_\rho \mathrm{e}^{\mathrm{i}kx} = \frac{\partial}{\partial x^\rho} \exp\left(\mathrm{i}k_\sigma x^\sigma\right) \tag{3.19}$$

$$= \exp\left(\mathrm{i}k_\sigma x^\sigma\right) \frac{\partial\left(\mathrm{i}k_\lambda x^\lambda\right)}{\partial x^\rho} \tag{3.20}$$

$$= \mathrm{i}k_\lambda \exp\left(\mathrm{i}k_\sigma x^\sigma\right) \frac{\partial x^\lambda}{\partial x^\rho} \tag{3.21}$$

$$= \mathrm{i}k_\lambda \exp\left(\mathrm{i}k_\sigma x^\sigma\right) \delta^\lambda_\rho \tag{3.22}$$

$$= \mathrm{i}k_\rho \exp\left(\mathrm{i}k_\sigma x^\sigma\right). \tag{3.23}$$

Thus,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu \mathrm{e}^{\mathrm{i}kx} = \eta^{\mu\nu}\,\mathrm{i}k_\mu\,\mathrm{i}k_\nu\,\mathrm{e}^{\mathrm{i}kx} \tag{3.24}$$

$$= -k_\mu k^\mu\,\mathrm{e}^{\mathrm{i}kx} \tag{3.25}$$

$$= -k^2\,\mathrm{e}^{\mathrm{i}kx}. \tag{3.26}$$

In conclusion,

$$(\Box + m^2)\phi(x) = \int \mathrm{d}^4k\,\tilde{\phi}(k)\,\left\{(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\mathrm{e}^{\mathrm{i}kx}\right\} \tag{3.27}$$

$$= \int \mathrm{d}^4k\,\tilde{\phi}(k)\,\left\{(-k^2 + m^2)\mathrm{e}^{\mathrm{i}kx}\right\} \tag{3.28}$$

$$= -\int \mathrm{d}^4k\,(k^2 - m^2)\,\tilde{\phi}(k)\,\mathrm{e}^{\mathrm{i}kx}. \tag{3.29}$$

Note that *on-shell* we have $(\Box + m^2)\phi(x) = 0$ hence, it holds $k^2 = \omega_{\boldsymbol{k}}^2 - \boldsymbol{k}^2 = m^2$, that is,

$$\omega_{\boldsymbol{k}}^2 = m^2 + \boldsymbol{k}^2, \tag{3.30}$$

for the particles that are on-shell. This can be used to split the general plane wave solution in the momentum space (obtained by the Fourier transform) into the positive and negative frequency parts (depending whether $k^0 > 0$ or $k^0 < 0$).

4 Ladder operators and Heisenberg Algebra

Useful Commutator Identities

$$[\,A,\,BC\,] \;=\; [\,A,B\,]\,C + B\,[\,A,C\,], \tag{4.1}$$

$$[\,AB,\,C\,] \;=\; A\,[\,B,C\,] + [\,A,C\,]\,B. \tag{4.2}$$

Quantum Harmonic Oscillator, reminder

(See also Section 1.2.2 Harmonic oscillator in Mandl and Shaw.)

We start from the Hamiltonian of the QHO,

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 \tag{4.3}$$

Now, we normalize the Hamiltonian $H' = H/\omega$ defining,

$$x' \to x\sqrt{m\omega}, \tag{4.4}$$
$$p' \to p/\sqrt{m\omega}, \tag{4.5}$$
$$H' \to H/\omega, \tag{4.6}$$

so that,

$$\frac{H}{\omega} = \frac{1}{2}\left(\frac{p}{\sqrt{m\omega}}\right)^2 + \frac{1}{2}\left(x\sqrt{m\omega}\right)^2, \tag{4.7}$$
$$H' = \frac{1}{2}p'^2 + \frac{1}{2}x'^2. \tag{4.8}$$

Since we have the canonical commutation relation,

$$\boxed{[x', p'] = [x, p] = \mathrm{i}}, \tag{4.9}$$

the Hamiltonian can not be diagonalized directly.

N.B. For the simplicity, from now on we continue with x' and p' without the primes!

We notice, however, that H is quadratic. It is easy to verify that by defining the ladder operators,

$$a = \frac{1}{\sqrt{2}}\left[x + \mathrm{i}p\right], \qquad \text{(the anihilation operator)} \tag{4.10}$$
$$a^\dagger = \frac{1}{\sqrt{2}}\left[x - \mathrm{i}p\right], \qquad \text{(the creation operator)} \tag{4.11}$$

we can complete the squares and obtain,

$$a^\dagger a = \frac{1}{\sqrt{2}}\left[x - \mathrm{i}p\right]\frac{1}{\sqrt{2}}\left[x + \mathrm{i}p\right] \tag{4.12}$$
$$= \frac{1}{2}\left[x^2 + p^2 + \mathrm{i}xp - \mathrm{i}px\right] \tag{4.13}$$
$$= \frac{1}{2}\left[x^2 + p^2 + \mathrm{i}[x, p]\right] \tag{4.14}$$
$$= \frac{1}{2}\left[x^2 + p^2\right] - \frac{1}{2}, \tag{4.15}$$

so that,

$$H = a^\dagger a + \frac{1}{2}, \tag{4.16}$$

or in the original unprimed variables $H = \omega\left(a^\dagger a + \frac{1}{2}\right)$.

It can be also verified that,

$$[a, a^\dagger] = \left[\frac{1}{\sqrt{2}}\left[x + \mathrm{i}p\right], \frac{1}{\sqrt{2}}\left[x - \mathrm{i}p\right]\right] = \frac{1}{2}\left[x, -\mathrm{i}p\right] + \frac{1}{2}\left[-\mathrm{i}p, x\right] = 1, \tag{4.17}$$

that is,

$$\boxed{[a, a^\dagger] = 1, \qquad aa^\dagger = 1 + a^\dagger a}. \tag{4.18}$$

This is an important property which is used to define the **Heisenberg algebra**.

We begin by introducing the number operator N,

$$N := a^\dagger a, \qquad \Rightarrow \qquad N = N^\dagger, \tag{4.19}$$

which means that N has real eigenvalues. Let λ be the eigenvalue for an eigenstate $|\lambda\rangle$,

$$N|\lambda\rangle = \lambda|\lambda\rangle, \tag{4.20}$$

then for $|\lambda'\rangle = a|\lambda\rangle$,

$$\langle\lambda|N|\lambda\rangle = \left\langle\lambda\middle|a^\dagger a\middle|\lambda\right\rangle = \lambda^2\langle\lambda|\lambda\rangle = \langle\lambda'|\lambda'\rangle \geq 0. \tag{4.21}$$

thus the spectra of N is non-negative.

We have also (see Problem 1),

$$[N, a] = -a, \qquad [N, a^\dagger] == a^\dagger. \tag{4.22}$$

Starting from $N|\lambda\rangle = \lambda|\lambda\rangle$ and the above relations,

$$N\left(a^\dagger|\lambda\rangle\right) = (\lambda + 1)|\lambda\rangle, \tag{4.23}$$

$$N(a|\lambda\rangle) = (\lambda - 1)|\lambda\rangle, \tag{4.24}$$

which means that a^n applied any eigenstate $|\lambda\rangle$ eventually becomes negative, so the maximum n is $n = \lfloor\lambda\rfloor$

$$a^{\lfloor\lambda\rfloor}|\lambda\rangle = (\lambda - \lfloor\lambda\rfloor)|\lambda\rangle, \tag{4.25}$$

otherwise, if we have one more a applied, the state will be with the negative eigenvalue. Thus we impose

$$a|\lambda - \lfloor\lambda\rfloor\rangle = 0, \tag{4.26}$$

to avoid the contradiction with negative eigenvalues (to flip-over from positive to negative; this will stop down-counting to eigenstate with eigenvalue 0). Hence we always have state $|0\rangle := |\lambda - \lfloor\lambda\rfloor\rangle$ with eigenvalue 0,

$$N|0\rangle = 0|0\rangle, \qquad a|0\rangle = 0. \tag{4.27}$$

All other states can be generated by using $a^\dagger$ as $(a^\dagger)^n|0\rangle$. We want states to be normalized to 1, so we define,

$$\langle 0|0\rangle := 1, \tag{4.28}$$

which gives,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|n\rangle\,. \tag{4.29}$$

5 Problems

Problem 1.

1. Evaluate,

$$[a^\dagger a, a], \qquad [a^\dagger a, a^\dagger], \qquad [a, (a^\dagger)^n]. \tag{5.1}$$

2. Prove,

$$[\, a,\ f(a^\dagger)\,] = \frac{\partial}{\partial a^\dagger} f(a^\dagger), \qquad [\, f(a),\ a^\dagger\,] = \frac{\partial}{\partial a} f(a) \tag{5.2}$$

Proposed solution
1a)

$$[a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger, a] a = -a \tag{5.3}$$

1b)

$$[a^\dagger a, a^\dagger] = a^\dagger [a, a^\dagger] + a^\dagger [a, a^\dagger] = a^\dagger \tag{5.4}$$

1c)

$$\begin{aligned} b_n &= [a, (a^\dagger)^n] = [a, a^\dagger (a^\dagger)^{n-1}] && (5.5)\\ &= [a, a^\dagger](a^\dagger)^{n-1} + a^\dagger [a, (a^\dagger)^{n-1}] && (5.6)\\ &= (a^\dagger)^{n-1} + a^\dagger b_{n-1} && (5.7)\\ &= (a^\dagger)^{n-1} + a^\dagger \left[(a^\dagger)^{n-2} + a^\dagger b_{n-2}\right] && (5.8)\\ &= n(a^\dagger)^{n-1} && (5.9) \end{aligned}$$

Proof by induction: $b_0 = [a, 1] = 0$, $b_1 = [a, a^\dagger] = 1$; then, assuming that

$$b_n = n(a^\dagger)^{n-1} \tag{5.10}$$

is valid for $n \geq 1$, using the reccurence relation, we derive b_{n+1},

$$\begin{aligned} b_{n+1} &= (a^\dagger)^n + a^\dagger b_n && (5.11)\\ &= (a^\dagger)^n + a^\dagger n (a^\dagger)^{n-1} && (5.12)\\ &= (n+1)(a^\dagger)^n, \text{ QED.} && (5.13) \end{aligned}$$

2a) Non-rigorous sketch (without bothering with spectra of $a^\dagger$), assuming $f(z)$ is analytic, with the power series expansion,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \tag{5.14}$$

we can use (1c) as

$$[\,a,\,f(a^\dagger)\,] = \left[a,\,\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(a^\dagger)^n\right] = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left[a,\,(a^\dagger)^n\right] \tag{5.15}$$

$$= \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}n(a^\dagger)^{n-1} = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{(n-1)!}(a^\dagger)^{n-1} \tag{5.16}$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n+1)}(0)}{n!}(a^\dagger)^n = \frac{\partial}{\partial a^\dagger}f(a^\dagger). \tag{5.17}$$

Similarly we derive (2b).

Problem 2. First show that,

$$a\,f(a^\dagger)\,|0\rangle = [a, f(a^\dagger)]\,|0\rangle\,, \tag{5.18}$$

$$a^n\,f(a^\dagger)\,|0\rangle = \frac{\partial^n}{\partial(a^\dagger)^n}f(a^\dagger)\,|0\rangle\,, \tag{5.19}$$

then calculate the following expectation values,

$$\left\langle 0\middle|a(a^\dagger)^2\middle|0\right\rangle, \qquad \left\langle 1\middle|a^\dagger a a^\dagger\middle|0\right\rangle, \qquad \langle n|x|n\rangle\,, \qquad \left\langle n\middle|x^2\middle|n\right\rangle, \qquad \left\langle 0\middle|a^n \mathrm{e}^{\alpha a^\dagger}\middle|0\right\rangle. \tag{5.20}$$

Proposed solution

Firstly, having $a\,|0\rangle = 0$,

$$a\,f(a^\dagger)\,|0\rangle = \left(af(a^\dagger) - f(a^\dagger)a\right)|0\rangle = [a, f(a^\dagger)]\,|0\rangle\,, \tag{5.21}$$

then applying recursively,

$$[\,a,\,f(a^\dagger)\,] = \frac{\partial}{\partial a^\dagger}f(a^\dagger), \tag{5.22}$$

we get,

$$a^n\,f(a^\dagger)\,|0\rangle = a\left(a^{n-1}f(a^\dagger)\right)|0\rangle = \frac{\partial^n}{\partial(a^\dagger)^n}f(a^\dagger)\,|0\rangle\,. \tag{5.23}$$

This yields the first two expectation values,

$$\left\langle 0\middle|a(a^\dagger)^2\middle|0\right\rangle = 2\left\langle 0\middle|a^\dagger\middle|0\right\rangle = 0, \tag{5.24}$$

and,

$$\left\langle 1\middle|a^\dagger a a^\dagger\middle|0\right\rangle = \left\langle 1\middle|a^\dagger a\middle|1\right\rangle = \langle 1|N|1\rangle = 1. \tag{5.25}$$

Note that we can also solve,

$$x = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right), \tag{5.26}$$

$$p = \frac{-\mathrm{i}}{\sqrt{2}}\left(a - a^\dagger\right), \tag{5.27}$$

so that,

$$\langle n|x|n\rangle = \left\langle 0 \middle| \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) \middle| 0 \right\rangle = 0, \tag{5.28}$$

and,

$$x^2 = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right)\frac{1}{\sqrt{2}}\left(a + a^\dagger\right) = \frac{1}{2}\left(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2\right), \tag{5.29}$$

$$\left\langle n \middle| x^2 \middle| n \right\rangle = \left\langle n \middle| \frac{1}{2}\left(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2\right) \middle| n \right\rangle \tag{5.30}$$

$$= \left\langle n \middle| \frac{1}{2}\left(\left\{1 + a^\dagger\right\} + a^\dagger a\right) \middle| n \right\rangle \tag{5.31}$$

$$= \left\langle n \middle| \left(N + \frac{1}{2}\right) \middle| n \right\rangle = N + \frac{1}{2} \tag{5.32}$$

Finally,

$$\left\langle 0 \middle| a^n \mathrm{e}^{\alpha a^\dagger} \middle| 0 \right\rangle = \left\langle 0 \middle| \alpha^n \mathrm{e}^{a^\dagger} \middle| 0 \right\rangle = \alpha^n, \tag{5.33}$$

$$\text{since} \quad \langle 0| \, \mathrm{e}^{a^\dagger} \, |0\rangle = \langle 0| \sum_{n=0}^{\infty} \frac{1}{n!}(a^\dagger)^n \, |0\rangle = \langle 0|0\rangle = 1. \tag{5.34}$$