Tutorial 3

2015-11-16 15:15 FB41 (Solutions)

Topics for today's and the oncoming tutorial

Necessary background. The Lagrangian of the real scalar (KGF) field. The quantization procedure. Also: going through the units, the normalization terms and the conventions for Fourier transform.

Part I. The inverse operator expansion, a and $a^\dagger$ as functions of ϕ and $\dot{\phi}$. The evaluation of the commutator $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]$. The evaluation of the equal time commutator $[\phi(x), \phi(y)]_{x^0=y^0=t}$).

Part II. The expansion of the Hamiltonian. An introduction to the infinite contributions to the energy and the normal ordering of operators.

Problems: [1]

For the real scalar field $\phi(x)$: (1) Evaluate the commutator $[a(\mathbf{k}), a^\dagger(\mathbf{k}')]$. (2) Evaluate $\partial_t\phi(x)$ and $\boldsymbol{\nabla}\phi(x)$. (3) Evaluate the commutator $[\phi(x), \pi(x')]$ at ET. (4) Expand the Hamiltonian in terms of the creation and annihilation operators.

The canonical quantization procedure (reminder)

- State the field.
  (The following examples are on the real scalar field. We also assume $\hbar = c = 1$.)

$$\phi(x) = \phi^*(x)$$

- State the Lagrangian density.

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi(x)\right)\left(\partial^\mu\phi(x)\right) - \frac{1}{2}m^2\phi^2(x)$$

- Vary the action

$$\delta S[\phi(x)] = \delta\int_\Omega \mathrm{d}^4x\,\mathcal{L}[\phi(x), \partial_\mu\phi(x)] = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0$$

  and derive the equations of motion,

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0$$

- Solve the equations of motion. (This is not always possible!)

$$\phi(x) = \boxed{\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a^*(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right],$$

$$k^0 = \omega_{\mathbf{k}} \equiv +\sqrt{|\mathbf{k}|^2 + m^2} = \omega_{-\mathbf{k}}$$

[1] We have very short time to solve the 'real' problems as, e.g., those given in the problem sets. What we can do instead is to highlight the used tools, techniques and common pitfails.

- In order to quantize this classical theory by the canonical formalism, we must introduce conjugate variables; thus go to Hamiltonian formalism and find $\phi(x)$ and $\pi(x)$.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(x)$$

- Promote $\phi(x)$ and $\pi(x)$ to operators. Observe that the complex conjugate a^* becomes the adjoint $\hat{a}^\dagger$!

$$\phi(x) \to \hat{\phi}(x), \qquad \pi(x) \to \hat{\pi}(x)$$

  We assume the hats further on and do not write them explicitly!

- Impose either the equal-time commutation relations (ETCR) for integer-spin fields, or the equal-time anticommutation relations (ETaCR) for spinor fields. Once we imposed ETCR/ETaCR, we get the quantum theory.

$$\left[\,\phi(x),\, \dot{\phi}(y)\,\right]_{x^0=y^0} = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y}),$$
$$[\,\phi(x),\, \phi(y)\,]_{x^0=y^0} = \left[\,\dot{\phi}(x),\, \dot{\phi}(y)\,\right]_{x^0=y^0} = 0$$

- The field operators will be expressed in terms of creation and annihilation operators (which will form a polynomial Heisenberg algebra for the integer-spin fields).

$$\left[\,a(\mathbf{k}),\, a^\dagger(\mathbf{k}')\,\right] = \delta_{\mathbf{k}\mathbf{k}'},$$
$$[\,a(\mathbf{k}),\, a(\mathbf{k}')\,] = \left[\,a^\dagger(\mathbf{k}),\, a^\dagger(\mathbf{k}')\,\right] = 0\,,$$
$$N(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k})$$

- From the symmetries of the action, find the conserved quantities ('charges' or 'constants of motion'), e.g., the Hamiltonian.

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\dot{\phi}^2(x) + (\boldsymbol{\nabla}\phi(x))\cdot(\boldsymbol{\nabla}\phi(x)) + m^2\phi^2(x)\right],$$
$$H = \int \mathrm{d}^3x\, \mathcal{H}(\phi, \pi, \partial_i\phi) = \frac{1}{2}\int \mathrm{d}^3x\, \left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi) + m^2\phi^2\right]$$

- To obtain the physical properties of particles, we express the constants of motion in terms of the creation and annihilation operators.

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right)$$

- We also need to handle the infinite expectation values for *operators* in the vacuum state. Before quantization we had any order of *functions* in equations. This is resolved by prescribing the **normal ordering** of operators. (This topic will be covered in later tutorials in more detail!) The normal ordering is usually done at the level of Lagrangian assuming the operators from beginning. **N.B.** In Mandl and Shaw, from pedagogical reasons, the normal ordering is not done at the level of Lagrangian for a real KGF! Compare M&S Eqs. (3.4) and (3.22).

1 Background, quantization of KGF

We consider a real scalar field,

$$\phi(x) = \phi^*(x)\,. \tag{1.1}$$

Such field corresponds to electrically neutral particles of spin-0 (e.g. K^0 or π^0 mesons). Charged particles are described by complex scalar fields.

The Lagrangian density for the real scalar field reads,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi(x)\right)\left(\partial^\nu \phi(x)\right) - \frac{1}{2}m^2\phi^2. \qquad \text{MS (3.4)}$$

From the stationary action principle, by varying action, we first obtain the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0, \tag{1.2}$$

which yields the equations of motion for the real scalar field,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -m^2\phi, && (1.3)\\
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\right] && (1.4)\\
&= \frac{1}{2}\partial_\mu\left[g^{\mu\sigma}\partial_\sigma\phi + g^{\rho\mu}\partial_\sigma\phi\right] && (1.5)\\
&= \partial_\mu\partial^\mu\phi, && (1.6)\\
\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} &= -m^2\phi - \partial_\mu\partial^\mu\phi = 0. && (1.7)
\end{aligned}$$

Hence we recovered the Klein-Gordon-Fock (KGF) equation,

$$\begin{aligned}
(\partial_\mu\partial^\mu + m^2)\phi(x) &= 0, && (1.8)\\
(\Box + m^2)\phi(x) &= 0. && \text{MS (3.3)}
\end{aligned}$$

The complete set of solutions of the KGF equation is,

$$\begin{aligned}
\phi(x) &= \phi^+(x) + \phi^-(x), && \text{MS (3.7a)}\\
\phi^+(x) &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx}, && \text{MS (3.7b)}\\
\phi^-(x) &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a^\dagger(\mathbf{k}) e^{ikx}, && \text{MS (3.7c)}\\
\phi(x) &= \boxed{\frac{1}{\sqrt{V}}\sum_{\mathbf{k}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right],
\end{aligned}$$

where k is the wave four-vector of a particle of mass m,

$$k^\mu k_\mu = m^2, \tag{1.9}$$

$$k^0 = \omega_{\mathbf{k}} = +\sqrt{|\mathbf{k}|^2 + m^2} = \omega_{-\mathbf{k}}. \tag{1.10}$$

In order to quantize this classical theory by the canonical formalism, we must introduce conjugate variables and then use the Legendre transformation. The momentum density $\pi(x)$ conjugated to $\phi(x)$ is easily obtained,

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \dot{\phi}(x). \qquad \text{MS (3.5)}$$

On quantization, the real field $\phi(x) = \phi^*(x)$ becomes a Hermitian operator, $\phi(x) = \phi^\dagger(x)$. Poisson bracket $\{\phi(x), \pi(x)\}$ is promoted to the equal time commutator $[\phi(x), \pi(y)]_{x^0=y^0}$, and we get the set of equal-time commutation relations,

$$\left[\phi(x),\, \dot{\phi}(y)\right]_{x^0=y^0} = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y}), \qquad \text{MS (3.6)}$$
$$[\,\phi(x),\, \phi(y)\,]_{x^0=y^0} = \left[\dot{\phi}(x),\, \dot{\phi}(y)\right]_{x^0=y^0} = 0\,.$$

From this, we arrive at the commutation relations,

$$\left[a(\mathbf{k}),\, a^\dagger(\mathbf{k}')\right] = \delta_{\mathbf{k}\mathbf{k}'}, \qquad \text{MS (3.9)}$$
$$[\,a(\mathbf{k}),\, a(\mathbf{k}')\,] = \left[a^\dagger(\mathbf{k}),\, a^\dagger(\mathbf{k}')\right] = 0\,.$$

These operators form an Heisenberg algebra[2], where,

$$N(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k}) \qquad \text{MS (3.10)}$$

have as their eigenvalues the occupation numbers (the operator $N(\mathbf{k})$ measures how many particles we have in a state),

$$n(\mathbf{k}) = 0,\, 1,\, 2,\, \ldots\,, \qquad \text{MS (3.11)}$$

while $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ are the annihilation and creation operators of particles with momentum $\mathbf{k}$ and energy $\omega_\mathbf{k}$.

Now, from the Noether's theorem we get the conserved quantities can be expressed in terms of the creation and annihilation operators.

For example, for the Hamiltonian,

$$H = \frac{1}{2}\int \mathrm{d}^3x \left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi) + m^2\phi^2\right] \qquad (1.11)$$
$$= \sum_{\mathbf{k}} \omega_\mathbf{k} \left[a^\dagger_\mathbf{k}(x) a_\mathbf{k}(x) + \frac{1}{2}\right]. \qquad (1.12)$$

[2]In fact, we get an infinite number of copies of the Heisenberg algebra (for each value of $\mathbf{k}$ we have one). This means that for each $\mathbf{k}$ we have a different the ground state $a(\mathbf{k})\,|0_\mathbf{k}\rangle = 0$, which further means that the vacuum is a continous tensor product,

$$|0\rangle = \bigotimes_{\mathbf{k}} |0_\mathbf{k}\rangle\,.$$

2 The inverse operator expansion

The complete set of solutions of the KGF equation (as in Eq. (3.7) M&S) but when integrating over the *continous momenta* is,

$$\phi(x) = \boxed{\int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right]$$

$$\frac{1}{\sqrt{V}}\sum_{\mathbf{k}} \sim \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3 k \qquad \text{cf. M\&S Eq. (1.48) on p. 12}$$

Note also (for our metric signature):

$$kx = k_\mu x^\mu = k_0 x^0 + k_i x^i = k^0 x^0 - \delta_{ij} k^i x^j = \omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x},$$

where $k_\mu k^\mu = m^2$ and,

$$k_0 = k^0 = \omega_{\mathbf{k}} := +\sqrt{|\mathbf{k}|^2 + m^2}; \qquad \omega_{\mathbf{k}} = \omega =_{-\mathbf{k}} > 0.$$

The conjugate field $\pi(x) = \partial_t \phi(x)$ is,

$$\begin{aligned}
\pi(x) &= \frac{\partial}{\partial t}\phi(t,\mathbf{x}) \\
&= \frac{\partial}{\partial t}\int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})}\right] \\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k})\frac{\partial}{\partial t}\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k})\frac{\partial}{\partial t}\mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})}\right] \\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a(\mathbf{k})\left(-\mathrm{i}\omega_{\mathbf{k}}\right)\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})} + a^\dagger(\mathbf{k})\left(\mathrm{i}\omega_{\mathbf{k}}\right)\mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x})}\right] \\
&= -\mathrm{i}\int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} - a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right].
\end{aligned}$$

Now,

$$\begin{aligned}
\int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}}\phi(t,\mathbf{x})\,\mathrm{e}^{\mathrm{i}kx} &= \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}}\mathrm{e}^{\mathrm{i}kx}\int \frac{\mathrm{d}^3 k'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k}')\mathrm{e}^{-\mathrm{i}k'x} + a^\dagger(\mathbf{k}')\mathrm{e}^{\mathrm{i}k'x}\right] \\
&= \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3 k'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k}')\mathrm{e}^{-\mathrm{i}k'x+\mathrm{i}kx} + a^\dagger(\mathbf{k}')\mathrm{e}^{\mathrm{i}k'x++\mathrm{i}kx}\right] \\
&= \int \frac{\mathrm{d}^3 k'}{(2\pi)^{3}} \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k}')\overbrace{\int \mathrm{d}^3 x\mathrm{e}^{-\mathrm{i}(k'-k)x}}^{(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')} + a^\dagger(\mathbf{k}')\overbrace{\int \mathrm{d}^3 x\mathrm{e}^{\mathrm{i}(k'+k)x}}^{(2\pi)^3\mathrm{e}^{\mathrm{i}2\omega_{\mathbf{k}}t}\delta^3(\mathbf{k}+\mathbf{k}')}\right] \\
&= \int \mathrm{d}^3 k' \frac{1}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a(\mathbf{k}')\delta^3(\mathbf{k}-\mathbf{k}') + a^\dagger(\mathbf{k}')\mathrm{e}^{\mathrm{i}2\omega_{\mathbf{k}}t}\delta^3(\mathbf{k}+\mathbf{k}')\right] \\
&= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}a(\mathbf{k}) + \frac{1}{\sqrt{2\omega_{-\mathbf{k}}}}a^\dagger(-\mathbf{k})\mathrm{e}^{\mathrm{i}2\omega_{-\mathbf{k}}t}.
\end{aligned}$$

$$\text{or:}\quad \boxed{(\mathrm{A})\qquad \int \frac{\mathrm{d}^3 x}{(2\pi)^{3/2}}\,\omega_{\pm\mathbf{k}}\phi(t,\mathbf{x})\,\mathrm{e}^{\mathrm{i}kx} = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}a(\mathbf{k}) + \sqrt{\frac{\omega_{-\mathbf{k}}}{2}}a^\dagger(-\mathbf{k})\mathrm{e}^{\mathrm{i}2\omega_{-\mathbf{k}}t}.}$$

Similarly,

$$\begin{aligned}
\int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\,\pi(t,\mathbf{x})\,\mathrm{e}^{\mathrm{i}kx} &= -\mathrm{i}\int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\,\mathrm{e}^{\mathrm{i}kx}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx} - a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}\right] \\
&= -\mathrm{i}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}a(\mathbf{k}) + \mathrm{i}\sqrt{\frac{\omega_{-\mathbf{k}}}{2}}a^\dagger(-\mathbf{k})\mathrm{e}^{\mathrm{i}2\omega_{-\mathbf{k}}t}.
\end{aligned}$$

$$\text{or:}\quad \boxed{(\mathrm{B})\quad \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\,\mathrm{i}\pi(t,\mathbf{x})\,\mathrm{e}^{\mathrm{i}kx} = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}a(\mathbf{k}) - \sqrt{\frac{\omega_{-\mathbf{k}}}{2}}a^\dagger(-\mathbf{k})\mathrm{e}^{\mathrm{i}2\omega_{-\mathbf{k}}t}\,.}$$

Solving for $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ from (A) and (B) yields,

$$\begin{aligned}
a(\mathbf{k}) &= \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}}\left[\omega_{\mathbf{k}}\phi(x) + \mathrm{i}\pi(x)\right], \\
a^\dagger(-\mathbf{k}) &= \mathrm{e}^{-\mathrm{i}2\omega_{-\mathbf{k}}t}\int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_{-\mathbf{k}}}}\left[\omega_{-\mathbf{k}}\phi(x) - \mathrm{i}\pi(x)\right].
\end{aligned}$$

We would like to get rid of $-\mathbf{k}$ and express everyting in terms of $\mathbf{k}$. We do this by change of variables,

$$\mathrm{e}^{-\mathrm{i}2\omega_{-\mathbf{k}}t}\mathrm{e}^{\mathrm{i}kx} \quad\overset{\mathbf{k}\to-\mathbf{k}}{\longrightarrow}\quad \mathrm{e}^{-\mathrm{i}2\omega_{\mathbf{k}}t}\mathrm{e}^{\mathrm{i}\omega_{-\mathbf{k}}t}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \mathrm{e}^{-\mathrm{i}kx}$$

Hence, we arrive at,

$$\begin{aligned}
a(\mathbf{k}) &= \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}}\left[\omega_{\mathbf{k}}\phi(x) + \mathrm{i}\pi(x)\right], \\
a^\dagger(\mathbf{k}) &= \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}}\frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}}\left[\omega_{\mathbf{k}}\phi(x) - \mathrm{i}\pi(x)\right],
\end{aligned}$$

which is in agreement that $a^\dagger(\mathbf{k}) = (a(\mathbf{k}))^\dagger$.

3 The commutator $[a(\mathbf{k}), a^\dagger(\mathbf{k}')]$

Let $x = (x^\mu) = (t, \mathbf{x})$ and $x' = (x'^\mu) = (t, \mathbf{y})$. We shall compute the commutator at equal-time $x^0 = x'^0 = t$,

$$\begin{aligned}
[a(\mathbf{k}), a^\dagger(\mathbf{k}')] &= \left[\int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} \frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_\mathbf{k}}} \left[\omega_\mathbf{k}\phi(x) + \mathrm{i}\pi(x)\right], \int \frac{\mathrm{d}^3y}{(2\pi)^{3/2}} \frac{\mathrm{e}^{-\mathrm{i}k'y}}{\sqrt{2\omega_{\mathbf{k}'}}} \left[\omega_{\mathbf{k}'}\phi(y) - \mathrm{i}\pi(y)\right]\right] \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \int \mathrm{d}^3y \frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_\mathbf{k}}} \frac{\mathrm{e}^{-\mathrm{i}k'y}}{\sqrt{2\omega_{\mathbf{k}'}}} \left[\left(\omega_\mathbf{k}\phi(x) + \mathrm{i}\pi(x)\right), \left(\omega_{\mathbf{k}'}\phi(y) - \mathrm{i}\pi(y)\right)\right] \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \int \mathrm{d}^3y \frac{\mathrm{e}^{\mathrm{i}kx - \mathrm{i}k'y}}{2\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} \Big\{\omega_\mathbf{k}\omega_{\mathbf{k}'}\cancel{[\phi(x), \phi(y)]} - \mathrm{i}\omega_\mathbf{k} \overbrace{[\phi(x), \pi(y)]}^{\mathrm{i}\delta^3(\mathbf{x}-\mathbf{y})} + \\
&\qquad + \mathrm{i}\omega_{\mathbf{k}'} \overbrace{[\pi(x), \phi(y)]}^{-\mathrm{i}\delta^3(\mathbf{y}-\mathbf{x})} - \mathrm{i}^2 \cancel{[\pi(x), \pi(y)]}\Big\} \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \int \mathrm{d}^3y \frac{\mathrm{e}^{\mathrm{i}kx - \mathrm{i}k'y}}{2\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} \Big\{\omega_\mathbf{k}\delta^3(\mathbf{x}-\mathbf{y}) + \omega_{\mathbf{k}'}\delta^3(\mathbf{y}-\mathbf{x})\Big\} \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \int \mathrm{d}^3y \frac{\mathrm{e}^{\mathrm{i}(\omega_\mathbf{k}t - \mathbf{k}\cdot\mathbf{x}) - \mathrm{i}(\omega_{\mathbf{k}'}t - \mathbf{k}'\cdot\mathbf{y})}}{2\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} \left(\omega_\mathbf{k} + \omega_{\mathbf{k}'}\right) \delta^3(\mathbf{x}-\mathbf{y}) \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \frac{\mathrm{e}^{\mathrm{i}\omega_\mathbf{k}t - \mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_{\mathbf{k}'}t + \mathrm{i}\mathbf{k}'\cdot\mathbf{x}}}{2\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} \left(\omega_\mathbf{k} + \omega_{\mathbf{k}'}\right) \\
&= \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \frac{\mathrm{e}^{\mathrm{i}(\omega_\mathbf{k} - \omega_{\mathbf{k}'})t}}{2\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} \left(\omega_\mathbf{k} + \omega_{\mathbf{k}'}\right) \mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \\
&= \frac{1}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}(\omega_\mathbf{k} - \omega_{\mathbf{k}'})t}}{2\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} \left(\omega_\mathbf{k} + \omega_{\mathbf{k}'}\right) (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \\
&= \delta^3(\mathbf{k}-\mathbf{k}')
\end{aligned}$$

Thus, finally,

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta^3(\mathbf{k}-\mathbf{k}'). \tag{3.1}$$

Cf. the discrete version M&S Eq. (3.9),

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}. \qquad \text{MS (3.9)}$$