Tutorial 5

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Topics for today

- Complex Scalar Field; Symmetries and Conservation Laws; Noether's theorem
- Representation of the Group, Group Parameters, Lie Groups
- The Rotation Group, SO(N); Unitary Groups, SU(N)

1 Problems

Problem 1. Show that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left((\partial\phi_1)^2 + (\partial\phi_2)^2\right) - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2 \tag{1.1}$$

is invariant under the transformation

$$\phi_1 \quad \to \quad \phi_1' = \phi_1 \cos\theta - \phi_2 \sin\theta, \tag{1.2}$$

$$\phi_2 \quad \to \quad \phi_2' = \phi_1 \sin\theta + \phi_2 \cos\theta. \tag{1.3}$$

Find the corresponding Noether current and charge.

Problem 2. The Lagrangian density of a real three-component scalar field is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\phi}^{\mathrm{T}}(x)\right)\left(\partial^\mu \boldsymbol{\phi}(x)\right) - \frac{m^2}{2}\boldsymbol{\phi}^{\mathrm{T}}(x)\boldsymbol{\phi}(x) \quad \text{where } \boldsymbol{\phi}(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \end{pmatrix}. \tag{1.4}$$

Find the equations of motions for the scalar fields $\phi_a(x)$. Prove that the Lagrangian density is SO(3) invariant and find the Noether currents.

Problem 3. Find the Euler-Langrange EoM for the following Lagrangian density

$$\mathcal{L} = \left(\partial_\mu \phi - \mathrm{i}eA_\mu \phi\right)\left(\partial_\mu \phi^* + \mathrm{i}eA^\mu \phi^*\right) - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.5}$$

Problem 4. Let $F_\mu = \partial_\mu \phi$ be a functional. Calculate the functional derivative $\frac{\delta F_\mu}{\delta \phi}$.

Problem 5. The Lagrangian density is given by[1]

$$\mathcal{L} = \bar{\boldsymbol{\psi}}(x)\left(\mathrm{i}\partial\!\!\!/ - m\right)\boldsymbol{\psi}(x), \quad \text{where } \boldsymbol{\psi}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. \tag{1.6}$$

Show that $\mathcal{L}$ has SU(2) symmetry. Find the Noether currents and charges. Derive the equations of motion for the spinor fields $\psi_a(x)$, where $a = 1, 2$.

[1] Herein the bold-faced $\boldsymbol{\psi}(x)$ is a doublet of the SU(2) group. The importance of this kind of expression will be obvious later in the course when studying electroweak theory.

2 Symmetries and Conservation Laws, recap

If $f(x)$ is a function and $F[f(x)]$ is a functional, the *functional derivative*, $\delta F/\delta f$ is defined by the relation

$$\delta F = \int \mathrm{d}y \boxed{\frac{\delta F[f(x)]}{\delta f(y)}} \delta f(y), \tag{2.1}$$

where δF is a variation of the functional.

The *action* is given by

$$S = \int \mathrm{d}^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{2.2}$$

where $\mathcal{L}$ is the Lagrangian density. From $\delta S = 0$, we get the *Euler-Lagrange equations of motion*

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{2.3}$$

The *cannonical momentum* conjugate to the field variable $\phi_a(x)$ is

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{2.4}$$

The *canonical Hamiltonian density* $\mathcal{H}$ and the Hamiltonian H are

$$\mathcal{H} = \dot{\phi}_a \pi^a - \mathcal{L}, \qquad H = \int \mathrm{d}^3x \mathcal{H} = \int \mathrm{d}^3x \left(\dot{\phi}_a \pi^a - \mathcal{L} \right). \tag{2.5}$$

Noether's theorem
states that if the action is invariant with respect to the continous infinitesimal transformations[2]

$$x_\mu \to x'_\mu = x_\mu + \delta_\epsilon x_\mu, \qquad \delta_\epsilon x^\mu = x'^\mu - x^\mu, \tag{2.6}$$

$$\phi_a(x) \to \phi'_a(x') = \phi_a(x) + \delta_\epsilon \phi_a(x), \qquad \delta_\epsilon \phi_a(x) = \phi'_a(x') - \phi_a(x), \tag{2.7}$$

$$\bar{\delta}_\epsilon \phi_a(x) = \phi'_a(x) - \phi_a(x), \tag{2.8}$$

then the divergence of the *Noether current* J^μ_ϵ is equal to zero $\partial_\mu J^\mu_\epsilon = 0$,

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L} \delta_\epsilon x^\mu = \sum_{r=1}^{d} \epsilon_r J^\mu_{\epsilon_r}, \tag{2.9}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_\epsilon \phi_a - T^\mu{}_\nu \delta_\epsilon x^\nu, \tag{2.10}$$

where $T^\mu{}_\nu$ is the *energy-momentum tensor*

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}\,. \tag{2.11}$$

The *Noether charges* Q_ϵ are constants of motion given by

$$Q_\epsilon = \int \mathrm{d}^3x\, J^0_\epsilon. \tag{2.12}$$

[2]The index ϵ is related to a symmetry group.
Note that $\bar{\delta}_\epsilon \phi_a(x)$ is calculated with both ϕ' and ϕ having the same argument x.