Tutorial 9

Topics for today

- The Schrödinger, Heisenberg and Interaction Pictures
- Perturbation Expansion of the S-Matrix
- Wick's Theorem

The Schrödinger, Heisenberg and Interaction Picture

	Schrödinger picture	**Heisenberg picture**	**Interaction picture**
State ket	Evolution determined by H	No change	Evolution determined by H_{int}
Observable	No change	Evolution determined by H	Evolution determined by H_0

Let $\mathcal{O}$ be a field operator (e.g. ϕ, ψ, A^μ...) and $|\Phi(t)\rangle^{\text{I}}$ be a state (the configuration of all the fields is inside $|\Phi(t)\rangle^{\text{I}}$!)

$$
\begin{gathered}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}^{\text{I}}(t) = [\mathcal{O}^{\text{I}}(t), H_0^{\text{I}}], \quad \text{equivalent to the free-field equations,} \\
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle^{\text{I}} = H^{\text{I}}_{\text{int}}(t)\,|\Phi(t)\rangle^{\text{I}}, \quad \text{governs evolution of states, (*)} \\
|\Phi(t_0)\rangle^{\text{S}} = |\Phi(t_0)\rangle^{\text{H}} = |\Phi(t_0)\rangle^{\text{I}}.
\end{gathered}
$$

At $t = t_0 \to -\infty$, we have the initial state $|i\rangle = |\Phi(-\infty)\rangle^{\text{I}}$.

At $t \to \infty$, we have all the possible final states as $|\Phi(\infty)\rangle^{\text{I}}$.

The solution, in general, can be written as:$|\Phi(\infty)\rangle^{\text{I}} = S\,|\Phi(-\infty)\rangle^{\text{I}}, \quad |\Phi(\infty)\rangle^{\text{I}} = S\,|i\rangle$,where S is obtained by solving (*).

Consider a scattering process from some initial particles $|i\rangle$, we get some final particles as $|f\rangle$. The projection of the final state $|f\rangle$ in the all possible final states is given by$\langle f|\Phi(t_0)\rangle$. This is *the contribution of* $|f\rangle$ *in* $|\Phi(\infty)\rangle^{\text{I}}$. The matrix S is called the scattering matrix, and has the matrix elements $S_{fi} = \langle f|S|i\rangle$.

The Lagrangian density of QED reads [with the **normal ordering** assumed!

$$
\mathcal{L} = \underbrace{\underbrace{\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi}_{\mathcal{L}_0^{\text{D}}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_0^{\text{M}}}}_{\mathcal{L}_0 = \mathcal{L}_0^{\text{D}} + \mathcal{L}_0^{\text{M}}} \underbrace{-q\bar{\psi}\gamma^\mu A_\mu\psi}_{\mathcal{L}_{\text{I}} = -s^\mu A_\mu}, \qquad \text{MS (6.9)}
$$

and the corresonding interaction Hamiltonian, with the normal ordering

$$
\mathcal{H}^{\text{I}}_{\text{int}} = q\,\mathrm{N}\left[\bar{\psi}\gamma^\mu A_\mu\psi\right]^{\text{I}} = \boxed{-e\,\mathrm{N}\left[\bar{\psi}\not{A}\psi\right]^{\text{I}}}. \qquad \text{MS (6.24)}
$$

Perturbation Expansion of the S-matrix

$$S = \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n \left[H_{\text{int}}^{\mathrm{I}}(t_1) H_{\text{int}}^{\mathrm{I}}(t_2) \ldots H_{\text{int}}^{\mathrm{I}}(t_n) \right] \qquad \text{MS (6.22a)}$$

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_n \mathrm{T} \left\{ H_{\text{int}}^{\mathrm{I}}(t_1) H_{\text{int}}^{\mathrm{I}}(t_2) \ldots H_{\text{int}}^{\mathrm{I}}(t_n) \right\} \qquad \text{MS (6.22b)}$$

The explicitly covariant S-matrix expansion, in terms of the interaction Hamiltonian density,

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}^4x_1 \int_{-\infty}^{\infty} \mathrm{d}^4x_2 \ldots \int_{-\infty}^{\infty} \mathrm{d}^4x_n \mathrm{T} \left\{ \mathcal{H}_{\mathrm{I}}(x_1) \mathcal{H}_{\mathrm{I}}(x_2) \ldots \mathcal{H}_{\mathrm{I}}(x_n) \right\} = \sum_{n=0}^{\infty} S^{(n)} \qquad \text{MS (6.23)}$$

$$S^{(n)} = \frac{\mathrm{i}^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}^4x_1 \int_{-\infty}^{\infty} \mathrm{d}^4x_2 \ldots \int_{-\infty}^{\infty} \mathrm{d}^4x_n \mathrm{T} \left\{ \mathcal{L}_{\text{int}}^{\mathrm{I}}(x_1) \mathcal{L}_{\text{int}}^{\mathrm{I}}(x_2) \ldots \mathcal{L}_{\text{int}}^{\mathrm{I}}(x_n) \right\}$$

Example, for QED:

$$\begin{aligned} \mathcal{H}_{\text{int}}^{\mathrm{I}}(x) &= -e \, \mathrm{N} \left[\bar{\psi}(x) \not{A}(x) \psi(x) \right]^{\mathrm{I}} \qquad \text{MS (7.2)} \\ &= -e \, \mathrm{N} \left[\left(\bar{\psi}^+ + \bar{\psi}^- \right)_x \left(\not{A}^+ + \not{A}^- \right)_x \left(\psi^+ + \psi^- \right)_x \right]^{\mathrm{I}} \end{aligned}$$

Table 1: Operators figuring in $\mathcal{H}_{\text{int}}^{\mathrm{I}}(x) = -e\mathrm{N}\left[\bar{\psi}(x)\not{A}(x)\psi(x)\right]^{\mathrm{I}}$

$A(x)$	$A^+(x)$	photon	absorption γ	$\varepsilon_r^\mu(\mathbf{k})$	$a_r(\mathbf{k})$	e^{-ikx}
	$A^-(x)$		creation γ	$\varepsilon_r^\mu(\mathbf{k})$	$a_r^\dagger(\mathbf{k})$	e^{ikx}
$\psi(x)$	$\psi^+(x)$	electron	absorption e^-	$u_{r\alpha}(\mathbf{p})$	$c_r(\mathbf{p})$	$\mathrm{e}^{-ipx/\hbar}$
	$\psi^-(x)$	positron	creation e^+	$v_{r\alpha}(\mathbf{p})$	$d_r^\dagger(\mathbf{p})$	$\mathrm{e}^{ipx/\hbar}$
$\bar{\psi}(x)$	$\bar{\psi}^+(x)$	positron	absorption e^+	$\bar{v}_{r\alpha}(\mathbf{p})$	$d_r(\mathbf{p})$	$\mathrm{e}^{-ipx/\hbar}$
	$\bar{\psi}^-(x)$	electron	creation e^-	$\bar{u}_{r\alpha}(\mathbf{p})$	$c_r^\dagger(\mathbf{p})$	$\mathrm{e}^{ipx/\hbar}$

Wick's Theorem

$$\mathrm{T}\left\{A(x)B(y)\right\} = \theta(x_0 - y_0) A(x) B(y) + \theta(y_0 - x_0) B(y) A(x)$$

$$\mathrm{N}(AB) = A^- B^+ + A^- B^- + A^+ B^+ + B^- A^+$$

$$AB = \mathrm{N}(AB) + \left[A^+, B^-\right]$$

$$\overbrace{AB} := \langle 0 | \mathrm{T}\left\{AB\right\} | 0 \rangle$$

$$\begin{aligned} \overbrace{\phi\phi} &= \mathrm{i}\Delta_{\mathrm{F}}, && \textbf{neutral scalar bosons}, \\ \overbrace{\phi^\star\phi} = \overbrace{\phi\phi^\star} &= \mathrm{i}\Delta_{\mathrm{F}}, && \textbf{charged scalar bosons}, \\ \overbrace{\psi\bar{\psi}} = -\overbrace{\bar{\psi}\psi} &= \mathrm{i}S_{\mathrm{F}}, && \textbf{fermions}, \\ \overbrace{AA} &= \mathrm{i}D_{\mathrm{F}}, && \textbf{vector bosons}. \end{aligned}$$