The Schrödinger, Heisenberg and Interaction Pictures in QFT (FK8017 HT15)

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Schrödinger Picture:

- **State kets are time-dependent** (governed by the Hamiltonian).
- Operators are stationary.
- Base kets are stationary.

Heisenberg Picture:

- State kets are stationary.
- **Operators are time-dependent** (governed by the Hamiltonian).
- Base kets are time-dependent (evolve in reverse wrt observables).

Schrödinger Equations of Motion (I)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,|\alpha;t\rangle^{\mathrm{S}} = H(t)\;|\alpha;t\rangle^{\mathrm{S}}$$
$$|\alpha;t\rangle^{\mathrm{S}}\,|_{t\to t_0} = |\alpha;t_0\rangle^{\mathrm{S}} = |\alpha\rangle^{\mathrm{H}}$$

Time-evolution (unitary) operator:

$$|\alpha;t\rangle^{\mathrm{S}} = \mathcal{U}(t,t_0)\;|\alpha;t_0\rangle^{\mathrm{S}}$$

Properties of $\mathcal{U}(t,t_0)$:

$$\mathcal{U}(t,t_0)^{\dagger}\mathcal{U}(t,t_0) = \mathbb{1}$$
$$\mathcal{U}(t_2,t_1) = \mathcal{U}(t_1,t_2)^{\dagger}$$
$$\mathcal{U}(t_3,t_2)\mathcal{U}(t_2,t_1) = \mathcal{U}(t_3,t_1)$$

Schrödinger Equations of Motion (II)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t,t_0) = H(t)\,\mathcal{U}(t,t_0) \qquad (1)$$
$$\mathcal{U}(t,t_0)|_{t\to t_0} = \mathbb{1}$$

The most general solution to EoM:

$$\mathcal{U}(t,t_0) = \mathrm{T}\left\{\exp\Big(-\,\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\,H(t')\Big)\right\} \qquad (2)$$

Solution when $\partial H/\partial t = 0$:

$$\mathcal{U}(t,t_0) = \exp\big(-\mathrm{i}H\,(t-t_0)\big)$$

Dyson series:

Solution to Schrödinger's EoM for any $H(t)$ where $H(t')$ and $H(t'')$ do not commute at different times $t'\neq t''$.

Rewrite (1) as the integral equation,

$$\int_{t_0}^{t}\mathrm{d}t'\,\mathcal{U}(t',t_0) = -\mathrm{i}\int_{t_0}^{t}\mathrm{d}t'\;H(t)\,\mathcal{U}(t',t_0),$$
$$\mathcal{U}(t,t_0) = \mathbb{1} - \mathrm{i}\int_{t_0}^{t}\mathrm{d}t_1\;H(t_1)\,\mathcal{U}(t_1,t_0).$$

Then reqursively expand $\mathcal{U}(t_1,t_0)$ in the integrand,

$$\mathcal{U}(t,t_0) = \mathbb{1} - \mathrm{i}\int_{t_0}^{t}\mathrm{d}t_1\;H(t)\,\mathcal{U}(t_1,t_0) + \ldots$$
$$+\,(-\mathrm{i})^n\int_{t_0}^{t}\mathrm{d}t_1\ldots\int_{t_0}^{t_{n-1}}\mathrm{d}t_n\;H(t_1)\cdots H(t_n) + \ldots$$

After introducing the time-ordering, we get,

$$\mathcal{U}(t,t_0) = \sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int_{t_0}^{t}\mathrm{d}t_1\ldots\int_{t_0}^{t}\mathrm{d}t_n\;\mathrm{T}\left\{H(t_1)\cdots H(t_n)\right\}$$

or in the condensed form, (2).

The S-matrix, definition:

$$S \equiv \mathcal{U}_{\mathrm{int}}(\infty,-\infty)$$

Expansion:

$$S = \sum_{n=0}^{\infty}S^{(n)},\ \text{ where } S^{(n)} \text{ is } n\text{-th order:}$$

$$\begin{aligned}S^{(n)} &= \frac{(-\mathrm{i})^n}{n!}\int_{-\infty}^{\infty}\mathrm{d}t_1\ldots\int_{-\infty}^{\infty}\mathrm{d}t_n\;\;\mathrm{T}\left\{H^{\mathrm{I}}_{\mathrm{int}}(t_1)\cdots H^{\mathrm{I}}_{\mathrm{int}}(t_n)\right\}\\ &= \frac{(-\mathrm{i})^n}{n!}\int\mathrm{d}x_1\ldots\int\mathrm{d}x_n\;\;\mathrm{T}\left\{\mathcal{H}^{\mathrm{I}}_{\mathrm{int}}(x_1)\cdots\mathcal{H}^{\mathrm{I}}_{\mathrm{int}}(x_n)\right\}\\ &= \frac{\mathrm{i}^n}{n!}\int\mathrm{d}x_1\ldots\int\mathrm{d}x_n\;\;\mathrm{T}\left\{\mathcal{L}^{\mathrm{I}}_{\mathrm{int}}(x_1)\cdots\mathcal{L}^{\mathrm{I}}_{\mathrm{int}}(x_n)\right\}\end{aligned}$$

where,

$$\mathcal{L}^{\mathrm{I}}_{\mathrm{int}}(x) := \mathcal{L}_{\mathrm{int}}[\phi^{\mathrm{I}}](x)$$

Covariance between S.P. and H.P.

For states:

$$|\alpha;t\rangle^{\mathrm{S}} = \mathcal{U}(t,t_0)\;|\alpha\rangle^{\mathrm{H}}$$
$$|\alpha;t_0\rangle^{\mathrm{S}} = |\alpha\rangle^{\mathrm{H}}$$

For operators:

$$A^{\mathrm{H}}(t) = \mathcal{U}^{\dagger}(t,t_0)\;A^{\mathrm{S}}\,\mathcal{U}(t,t_0)$$
$$A^{\mathrm{H}}(t_0) = A^{\mathrm{S}}$$

The last holds also for the Hamiltonian H:

$$H^{\mathrm{H}}(t) = \mathcal{U}^{\dagger}(t,t_0)\;H^{\mathrm{S}}\,\mathcal{U}(t,t_0)$$
$$H^{\mathrm{H}}(t_0) = H^{\mathrm{S}}$$

If $\partial H/\partial t = 0$ then $H^{\mathrm{H}} = H^{\mathrm{S}} \equiv H$.

Heisenberg Equations of Motion

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{H}}(t) = \left[A^{\mathrm{H}}(t),H^{\mathrm{H}}(t)\right] + \frac{\partial}{\partial t}A^{\mathrm{H}}(t)$$
$$A^{\mathrm{H}}(t)|_{t\to t_0} = A^{\mathrm{H}}(t_0) = A^{\mathrm{S}}$$

Example of states (in Fock space):

$$|n_{\mathbf{k}}\rangle^{\mathrm{H}} = \frac{1}{\sqrt{n!}}\left(a^{\dagger}(\mathbf{k})\right)^{n}|0\rangle^{\mathrm{H}}$$
$$|1_{\mathbf{p},r}\rangle^{\mathrm{H}} = |\mathrm{e}^-,\mathbf{p},r\rangle^{\mathrm{H}} = c_r^{\dagger}(\mathbf{p})\,|0\rangle^{\mathrm{H}}$$
$$|1_{\mathbf{p_1},r_1};\,\bar{1}_{\mathbf{p_2},r_2}\rangle^{\mathrm{H}} = |\mathrm{e}^-,\mathbf{p_1},r_1;\,\mathrm{e}^+,\mathbf{p_2},r_2\rangle^{\mathrm{H}}$$

Example of operators:

$$\phi(\mathbf{x})^{\mathrm{S}},a^{\mathrm{S}}(\mathbf{k}),c_r^{\dagger}(\mathbf{p})$$
$$\phi(\mathbf{x},t)^{\mathrm{H}} = \mathcal{U}^{\dagger}(t)\,\phi(\mathbf{x})^{\mathrm{S}}\,\mathcal{U}(t)$$

Example of Heisenberg EoM. Starting from,

$$\phi(\mathbf{x})^{\mathrm{S}} = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\Big(a^{\mathrm{S}}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + a^{\dagger\,\mathrm{S}}(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\Big),$$
$$\phi(\mathbf{x},t)^{\mathrm{H}} = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\Big(a^{\mathrm{H}}(\mathbf{k},t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} + a^{\dagger\,\mathrm{H}}(\mathbf{k},t)\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\Big),$$
$$H^{\mathrm{S}} = \sum_{\mathbf{k}}\omega_{\mathbf{k}}\;a^{\dagger\,\mathrm{S}}(\mathbf{k})\,a^{\mathrm{S}}(\mathbf{k}).$$

The EoM reads,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a^{\mathrm{H}}(\mathbf{k},t) = \left[a^{\mathrm{H}}(\mathbf{k},t),H^{\mathrm{H}}(t)\right],$$
$$\text{initial cond.:}\quad a^{\mathrm{H}}(\mathbf{k},t=0) = a^{\mathrm{S}}(\mathbf{k}),$$

where,

$$a^{\mathrm{H}}(\mathbf{k},t) = \mathcal{U}^{\dagger}(t)\,a^{\mathrm{S}}(\mathbf{k})\,\mathcal{U}(t),$$
$$H^{\mathrm{H}}(t) = \mathcal{U}^{\dagger}(t)\;H^{\mathrm{S}}\,\mathcal{U}(t),$$
$$\mathcal{U}(t) = \exp\left(-\mathrm{i}H^{\mathrm{S}}t\right).$$

Using $[H^{\mathrm{S}},a^{\mathrm{S}}(\mathbf{k})] = -\omega_{\mathbf{k}}\,a^{\mathrm{S}}(\mathbf{k})$, the EoM becomes,

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}a^{\mathrm{H}}(\mathbf{k},t) = \omega_{\mathbf{k}}\,a^{\mathrm{H}}(\mathbf{k},t),$$

with the solution (note that $H\equiv H^{\mathrm{S}} = H^{\mathrm{H}}$),

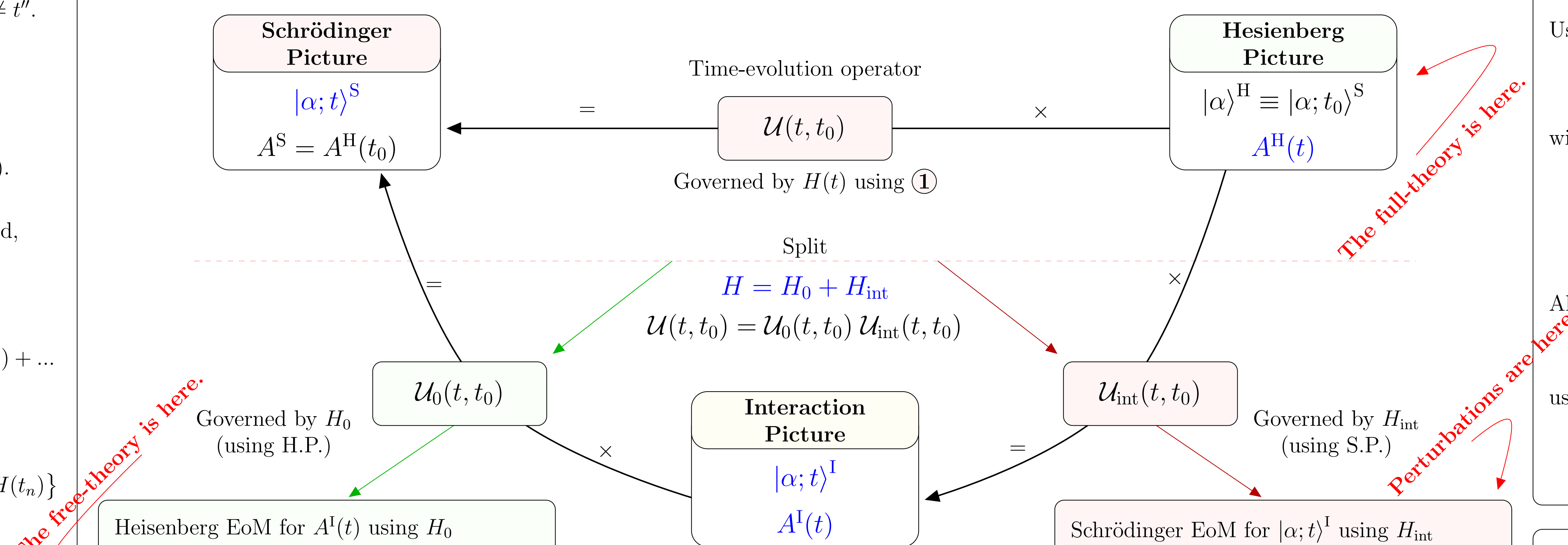
$$a^{\mathrm{H}}(\mathbf{k},t) = a^{\mathrm{H}}(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t},$$
$$\phi(\mathbf{x},t)^{\mathrm{H}} = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\Big(a^{\mathrm{S}}(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}kx} + a^{\dagger\,\mathrm{S}}(\mathbf{k})\,\mathrm{e}^{\mathrm{i}kx}\Big).$$

Alternatively, we can find $\phi^{\mathrm{H}}(x)$ directly from,

$$\phi(\mathbf{x},t)^{\mathrm{H}} = \mathrm{e}^{\mathrm{i}Ht}\;\phi(\mathbf{x})^{\mathrm{S}}\;\mathrm{e}^{-\mathrm{i}Ht},$$

using the Baker-Campbell-Hausdorff formula,

$$\mathrm{e}^{A}\,B\,\mathrm{e}^{-A} = B + [A,B] + \tfrac{1}{2!}[A,[A,B]] + \tfrac{1}{n!}[A,[A,\cdots,[A,B]\,]] + \ldots.$$



Heisenberg EoM for $A^{\mathrm{I}}(t)$ using H_0

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{I}}(t) = \left[A^{\mathrm{I}}(t),\,\boxed{H_0^{\mathrm{I}}(t)}\,\right]$$
$$A^{\mathrm{I}}(t_0) = A^{\mathrm{H}}(t_0) = A^{\mathrm{S}}$$

In the Interaction Picture, the fields ϕ^{I} retain the properties of the free fields. (The fields ϕ^{I} are the solutions to the free-theory EoM for H_0 obtained from $\mathcal{L}_0$.)

We postulate that the canonical commuatation relations are valid also for the *interacting field operators* with no gradient couplings that modify the conjugate field (i.e., when the interaction Lagrangian $\mathcal{L}_{\mathrm{int}}$ does not contain derivatives wrt fields).

At any fixed time, the full-interacting creator and annihilation operators satisfy the same algebra as in the free-theory (due to translation invariance of the Fock space).

Covariance between S.P., I.P. and H.P.

For states:

$$|\alpha;t\rangle^{\mathrm{S}} = \mathcal{U}_0(t,t_0)\;|\alpha;t\rangle^{\mathrm{I}}$$
$$|\alpha;t\rangle^{\mathrm{I}} = \mathcal{U}_{\mathrm{int}}(t,t_0)\;|\alpha\rangle^{\mathrm{H}}$$
$$|\alpha;t\rangle^{\mathrm{S}} = \mathcal{U}_0(t,t_0)\,\mathcal{U}_{\mathrm{int}}(t,t_0)\;|\alpha\rangle^{\mathrm{H}}$$
$$|\alpha;t_0\rangle^{\mathrm{S}} = |\alpha;t_0\rangle^{\mathrm{I}} = |\alpha\rangle^{\mathrm{H}}$$

For operators:

$$A^{\mathrm{I}}(t) = \mathcal{U}_0^{\dagger}(t,t_0)\;A^{\mathrm{S}}\,\mathcal{U}_0(t,t_0)$$
$$A^{\mathrm{H}}(t_0) = A^{\mathrm{I}}(t_0) = A^{\mathrm{S}}$$

Note:

$$H^{\mathrm{I}}_{\mathrm{int}}(t) = \mathcal{U}_0^{\dagger}(t,t_0)\;H^{\mathrm{S}}_{\mathrm{int}}\,\mathcal{U}_0(t,t_0)$$
$$H^{\mathrm{I}}_0(t) = \mathcal{U}_0^{\dagger}(t,t_0)\;H^{\mathrm{S}}_0\,\mathcal{U}_0(t,t_0)$$
$$H^{\mathrm{I}}_0 = H^{\mathrm{I}}_0(t_0) = H^{\mathrm{S}}_0$$

Schrödinger EoM for $|\alpha;t\rangle^{\mathrm{I}}$ using H_{int}

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}_{\mathrm{int}}(t,t_0) = \boxed{H_{\mathrm{int}}(t)}\,\mathcal{U}_{\mathrm{int}}(t,t_0)$$
$$\mathcal{U}_{\mathrm{int}}(t,t_0)|_{t\to t_0} = \mathbb{1}$$

Limit I.P. → H.P.

In the limit,

$$H_{\mathrm{int}} \to 0$$

we have,

$$\begin{aligned}H &\to H_0 & |\alpha;t\rangle^{\mathrm{I}} &\to |\alpha\rangle^{\mathrm{H}}\\ \mathcal{U}_{\mathrm{int}}(t) &\to \mathbb{1} & \phi^{\mathrm{I}}(x) &\to \phi^{\mathrm{H}}(x)\\ \mathcal{U}(t) &\to \mathcal{U}_0(t) & &\end{aligned}$$

We can replace "I" by "H" in all the expressions (this is the limit when the theory becomes free). This also means that, by introducting H_{int}, we have moved the operators from H.P. (solving the free-theory) to I.P.

Wick's theorem is a method of expanding the time-ordered products in the S-matrix as a sum of normal products. It exploits a similar behaviour of the time-ordering T{} and the normal-ordering N{} meta operators. Namely, they (i) both treat boson/fermions equally, and (ii) both suppress equal-time (anti)commutation relations.

For two boson operators, the following relation holds:

$$AB = \mathrm{N}\,(AB) + \left[A^+,B^-\right]$$

For two fermion operators we have anti-commutator instead. The last object is a c-number and becomes the propagator when time-ordered. Wick's theorem states that, at unequal-times, for any two operators it holds,

$$\mathrm{T}\,\{AB\} = \mathrm{N}\,\{AB\} + \langle 0|AB|0\rangle$$

The last term is the so called **contraction** between the fields. The contractions are always between virtual (off-shell) particles and never observed.