Casimir effect

(Extra material to Tutorial 4)

Here we investigate consequences of the vacuum energy on a **scalar boson**. (We usually think of the electromagnetic field when we think of the Casimir energy, but the argument works as well for a scalar field!)

Let's imagine a massless ($m = 0$) scalar boson in one dimension. The energy of this boson is then $E_n = c\,|p_n|$ where $p_n = \hbar\frac{2\pi}{L}n$ where L is the length of the system. Thus $E_n \propto \frac{n}{L}$. Even if there are no particles present, the vacuum will contribute with energy $\frac{E_n}{2}$ for each mode in the system. The vacuum energy can be modeled using the thermodynamic partition function as

$$Z\left(\beta\right) = \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\frac{\beta E_n}{2}}.$$

The vacuum energy contribution is $E\left(L\right) = -\left.\partial_\beta Z\right|_{\beta=0} = \frac{1}{2}\sum_{m=-\infty}^{\infty} E_n$, and is of course infinite. Now we are interested in changes to this infinite energy as L is changed, therefore we compute Z. For simplicity we write $\frac{E_n}{2} = E_L\,|n|$ and get

$$\begin{aligned}
Z\left(\beta\right) &= \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\beta E_L |n|} = 2\sum_{m=0}^{\infty} \mathrm{e}^{-\beta E_L n} - 1 \\
\text{(geometric series)} &= 2\sum_{m=0}^{\infty}\left(\mathrm{e}^{-\beta E_L}\right)^n - 1 = \frac{2}{1-\mathrm{e}^{-\beta E_L}} - 1 \\
\text{(rewrite)} &= \frac{2\mathrm{e}^{\frac{\beta E_L}{2}}}{\mathrm{e}^{\frac{\beta E_L}{2}} - \mathrm{e}^{-\frac{\beta E_L}{2}}} - \frac{\mathrm{e}^{\frac{\beta E_L}{2}} - \mathrm{e}^{-\frac{\beta E_L}{2}}}{\mathrm{e}^{\frac{\beta E_L}{2}} - \mathrm{e}^{-\frac{\beta E_L}{2}}} = \frac{\mathrm{e}^{\frac{\beta E_L}{2}} + \mathrm{e}^{-\frac{\beta E_L}{2}}}{\mathrm{e}^{\frac{\beta E_L}{2}} - \mathrm{e}^{-\frac{\beta E_L}{2}}} \\
&= \frac{\cosh\frac{\beta E_L}{2}}{\sinh\frac{\beta E_L}{2}}.
\end{aligned}$$

To compute the energy we need the derivative with respect to β. Now, remember that the derivatives are $\partial_x \sinh x = \cosh x$ and $\partial_x \cosh x = \sinh x$. Then we have

$$\begin{aligned}
E\left(\beta, L\right) &= -\partial_\beta Z\left(\beta\right) = -\partial_\beta \frac{\cosh\frac{\beta E_L}{2}}{\sinh\frac{\beta E_L}{2}} \\
&= \frac{E_L}{2}\frac{\cosh^2\frac{\beta E_L}{2} - \sinh^2\frac{\beta E_L}{2}}{\sinh^2\frac{\beta E_L}{2}} \\
&= \frac{E_L}{2\sinh^2\frac{\beta E_L}{2}}.
\end{aligned}$$

Now comes the crucial step: Imagine that we split the 1-D world into two partitions, one of length ℓ and one of length $L-\ell$ where $\ell \ll L$. The point here is that

$$E\left(L\right) \neq E\left(\ell\right) + E\left(L-\ell\right),$$

but also that

$$\partial_\ell\left(E\left(\ell\right) + E\left(L-\ell\right)\right) \neq 0,$$

such that the energy depends on the size of ℓ. Of our particular interest is the difference between

$E(L)$ and $E(\ell)+E(L-\ell)$, we therefore compute

$$\begin{aligned}\Delta(\beta,L,\ell) &= E(\beta,\ell)+E(\beta,L-\ell)-E(\beta,L)\\ &= \frac{E_\ell}{2\sinh^2\frac{\beta E_\ell}{2}}+\frac{E_{L-\ell}}{2\sinh^2\frac{\beta E_{L-\ell}}{2}}+\frac{E_L}{2\sinh^2\frac{\beta E_L}{2}}.\end{aligned}$$

We use that for small x, $\sinh x\approx x$. This is always valid since $\beta\to 0$ at the end. Consequently

$$\begin{aligned}\Delta(\beta,L,\ell) &= \frac{E_\ell}{2\left(\frac{\beta E_\ell}{2}\right)^2}+\frac{E_{L-\ell}}{2\left(\frac{\beta E_{L-\ell}}{2}\right)^2}-\frac{E_L}{2\left(\frac{\beta E_L}{2}\right)^\ell}+\mathcal{O}\left(\beta^{-1}\right)\\ &= \frac{2}{\beta^2 E_\ell}+\frac{2}{\beta^2 E_{L-\ell}}-\frac{2}{\beta^2 E_L}+\mathcal{O}\left(\beta^{-1}\right)\\ &= 0+\mathcal{O}\left(\beta^{-1}\right).\end{aligned}$$

Hence we need to expand to one further order: We then have $\sinh^2 x\approx x^2+\frac{1}{3}x^4$. Expanding this yields

$$\Delta(\beta,L,\ell) \approx -E_\ell\cdot\frac{\ell^2-\ell L+L^2}{3(L-\ell)L}+\mathcal{O}(\beta)\overset{\ell\ll L}{\approx}-\frac{E_\ell}{3}+\mathcal{O}(\beta).$$

Notice that β is not part of the difference in the leading order. We may therefore safely put $\beta=0$ in the expression. Finally if we expand $E_\ell=c\hbar\frac{2\pi}{\ell}$ we see that the energy gained by introducing a partition is

$$\Delta(\ell)=-2\pi\frac{c\hbar}{3}\cdot\frac{1}{\ell},$$

and is negative and increasingly so for smaller ℓ. Converting energy into a force gives

$$F(\ell)=-\partial_\ell\Delta(\ell)=-2\pi\frac{c\hbar}{3}\cdot\frac{1}{\ell^2},$$

such that the force pushing the plates together becomes stronger and stronger as the plates come close together. This is the Casimir effect.