Tutorial 12

Topics for today

- P1: Weyl fields (and problem of coupling by mass)
- P2: Transformation properties of vectors/axial vectors ☼
- P3: Massive vector bosons (the Proca equation)
- P4: EoM for the Yang-Mills Lagrangian ☼
- P5: SU(2) charges of weak interactions

Problem 1

Consider

$$\begin{aligned}\psi_{\mathrm{L}} &= \frac{1}{2}(1-\gamma_5)\psi,\\ \psi_{\mathrm{R}} &= \frac{1}{2}(1+\gamma_5)\psi,\end{aligned}$$

where ψ is a Dirac spinor. Derive the equations of motion for these fields. Show that they are decoupled in the case of a massless spinor. (These fields ψ_{L} and ψ_{R} are known as Weyl fields.)

Proposed Solution

By multiplying the Dirac equation from the left by γ_5, we obtain

$$\begin{aligned}\gamma_5(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi &= 0,\\ (\mathrm{i}\gamma^\mu\gamma_5\partial_\mu + \gamma_5 m)\psi &= 0,\\ (\mathrm{i}\partial\!\!\!/ + m)\gamma_5\psi &= 0.\end{aligned}$$

Now apply $\mathrm{i}\partial\!\!\!/$ and m (multiply by) on ψ_{L} and ψ_{R} from the left

$$\mathrm{i}\partial\!\!\!/\psi_{\mathrm{L}} = \mathrm{i}\partial\!\!\!/\left[\frac{1}{2}(1-\gamma_5)\psi\right] = \frac{1}{2}\left[\mathrm{i}\partial\!\!\!/\psi - \mathrm{i}\partial\!\!\!/\gamma_5\psi\right] \tag{1}$$

$$\mathrm{i}\partial\!\!\!/\psi_{\mathrm{R}} = \mathrm{i}\partial\!\!\!/\left[\frac{1}{2}(1+\gamma_5)\psi\right] = \frac{1}{2}\left[\mathrm{i}\partial\!\!\!/\psi + \mathrm{i}\partial\!\!\!/\gamma_5\psi\right] \tag{2}$$

$$m\psi_{\mathrm{L}} = \frac{1}{2}(m\psi - m\gamma_5\psi) \tag{3}$$

$$m\psi_{\mathrm{R}} = \frac{1}{2}(m\psi + m\gamma_5\psi) \tag{4}$$

Subtract (4) from (1) and (3) from (2)

$$\begin{aligned}\mathrm{i}\partial\!\!\!/\psi_{\mathrm{L}} - m\psi_{\mathrm{R}} &= \frac{1}{2}\left[\mathrm{i}\partial\!\!\!/\psi - \mathrm{i}\partial\!\!\!/\gamma_5\psi - m\psi - m\gamma_5\psi\right]\\ &= \frac{1}{2}\left[\Big(\mathrm{i}\partial\!\!\!/\psi - m\psi\Big) - \Big(\mathrm{i}\partial\!\!\!/\gamma_5\psi + m\gamma_5\psi\Big)\right]\\ \mathrm{i}\partial\!\!\!/\psi_{\mathrm{R}} - m\psi_{\mathrm{L}} &= \frac{1}{2}\left[\mathrm{i}\partial\!\!\!/\psi + \mathrm{i}\partial\!\!\!/\gamma_5\psi - m\psi + m\gamma_5\psi\right]\\ &= \frac{1}{2}\left[\Big(\mathrm{i}\partial\!\!\!/\psi - m\psi\Big) + \Big(\mathrm{i}\partial\!\!\!/\gamma_5\psi + m\gamma_5\psi\Big)\right]\end{aligned}$$

thus

$$\begin{aligned}\mathrm{i}\partial\!\!\!/\psi_{\mathrm{L}} - m\psi_{\mathrm{R}} &= 0,\\ \mathrm{i}\partial\!\!\!/\psi_{\mathrm{R}} - m\psi_{\mathrm{L}} &= 0.\end{aligned}$$

Clearly, the fields are decoupled in the case of a massless spinor.

Problem 2 ☼

For a Dirac field ψ, obtain the action of Lorentz transformations on ψ. Investigate the transformation properties under proper orthochronous Lorentz transformation of:

- $\bar{\psi}\psi$,
- $\bar{\psi}\gamma^\mu\psi$,
- $\bar{\psi}\gamma^\mu\gamma^5\psi$.

Assumptions

Let Λ be an ocrthochronous Lorentz transformations,

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \text{MS (A.48)}$$

i.e. $\Lambda^0{}_0 > 0$ and $\det(\Lambda^\mu{}_\nu) = \pm 1$, so that the sense of time is not reversed, but the transformation may or may not involve spatial inversion.

It can be shown[1] corresponding that to each such Lorentz transformation Λ one can construct a non-singular 4x4 matrix $S = S(\Lambda)$ with the properties

$$\gamma^\nu = \Lambda^\nu{}_\mu S\gamma^\mu S^{-1} \qquad \text{MS (A.49)}$$

and

$$S^{-1} = \gamma^0 S^\dagger \gamma^0. \qquad \text{MS (A.50)}$$

If the transformation properties of the Dirac spinor $\psi(x)$ are defined by

$$\psi(x) \to \psi'(x') = S\psi(x), \qquad \text{MS (A.52)}$$

we have the transformation properties of the corresponding adjoint spinor

$$\bar{\psi}(x) \to \bar{\psi}'(x') = (S\psi(x))^\dagger \gamma^0 = \psi^\dagger(x)S^\dagger\gamma^0 = \psi^\dagger(x)\gamma^0 S^{-1} = \bar{\psi}(x)S^{-1}.$$

Proposed Solution

(1) For the first case $\bar{\psi}\psi$, we have (see Tutorial 8)

$$\bar{\psi}\psi \to \bar{\psi}'\psi' = \bar{\psi}S^{-1}S\psi = \bar{\psi}\psi,$$

thus $\bar{\psi}\psi$ is invariant under the Lorentz gransformations and behaves as a **scalar**.

[1]For its derivation Mandl and Shaw referenced pp. 358-359 in H. A. Bethe and R. W. Jackiw, *Intermediate Quantum Mechanics*, 2nd edn, Benjamin, New York, 1968.

(2) For the case $\bar{\psi}\gamma^\mu\psi$ we get,

$$\begin{aligned}\bar{\psi}\gamma^\mu\psi \to \bar{\psi}'\gamma^\mu\psi' &= \left(\bar{\psi}S^{-1}\right)\gamma^\mu\left(S\psi\right)\\ &= \bar{\psi}\left(S^{-1}\gamma^\mu S\right)\psi\end{aligned}$$

On the other hand, (note that $\Lambda^\mu{}_\nu$ are just numbers)

$$\begin{aligned}\gamma^\mu &= \Lambda^\mu{}_\nu S\gamma^\nu S^{-1}\\ \gamma^\mu &= S\Lambda^\mu{}_\nu\gamma^\nu S^{-1}\\ S^{-1}\gamma^\mu S &= \Lambda^\mu{}_\nu\gamma^\nu\end{aligned}$$

Substituting in the previous expression yields,

$$\begin{aligned}\bar{\psi}'\gamma^\mu\psi' &= \bar{\psi}\left(\Lambda^\mu{}_\nu\gamma^\nu\right)\psi\\ &= \Lambda^\mu{}_\nu\left(\bar{\psi}\gamma^\nu\psi\right)\end{aligned}$$

and since vectors transform as $V'^\mu = \Lambda^\mu{}_\nu V^\nu$ we conclude that $\bar{\psi}\gamma^\nu\psi$ transform as a **vector**.

(3) For the last case $\bar{\psi}\gamma^\mu\gamma^5\psi \to \bar{\psi}'\gamma^\mu\gamma^5\psi'$, we obtain,

$$\begin{aligned}\bar{\psi}'\gamma^\mu\gamma^5\psi' &= \left(\bar{\psi}S^{-1}\right)\gamma^\mu\gamma^5\left(S\psi\right)\\ &= \bar{\psi}\left(S^{-1}\gamma^\mu SS^{-1}\gamma^5 S\right)\psi\\ &= \bar{\psi}\left(S^{-1}\gamma^\mu S\right)\left(S^{-1}\gamma^5 S\right)\psi\\ & \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\left(S^{-1}\gamma^5 S\right)\psi\end{aligned}$$

and here starts the fun stuff, to simplify $S^{-1}\gamma^5 S$. We start from the definition of γ^5,

$$\gamma^5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 \qquad \text{MS (A.7)}$$

rewritting γ^5 using the completely antisymmetric alternating symbol $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$ which is equal to $+1$ for $(\mu_1\mu_2\mu_3\mu_4)$ an even permutation of $(0, 1, 2, 3)$, is equal to -1 for an odd permutation, and vanishes if two or more indices are the same.

$$\gamma^5 = \frac{\mathrm{i}}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}.$$

This gives,

$$\begin{aligned}S^{-1}\gamma^5 S &= \frac{\mathrm{i}}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}S\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}S^{-1}\\ &= \frac{\mathrm{i}}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}S\gamma^{\mu_1}S^{-1}\,S\gamma^{\mu_2}S^{-1}\,S\gamma^{\mu_3}S^{-1}\,S\gamma^{\mu_4}S^{-1}\\ &= \frac{\mathrm{i}}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\Lambda^{\mu_1}{}_{\nu_1}\gamma^{\nu_1}\,\Lambda^{\mu_2}{}_{\nu_2}\gamma^{\nu_2}\,\Lambda^{\mu_3}{}_{\nu_3}\gamma^{\nu_3}\,\Lambda^{\mu_4}{}_{\nu_4}\gamma^{\nu_4}\end{aligned}$$

$$= \frac{\mathrm{i}}{4!}\epsilon_{\mu_1\mu_2\mu_3\mu_4}\Lambda^{\mu_1}{}_{\nu_1}\Lambda^{\mu_2}{}_{\nu_2}\Lambda^{\mu_3}{}_{\nu_3}\Lambda^{\mu_4}{}_{\nu_4}\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4}$$

Here we recognize an expression for the determinant of Λ

$$\det(\Lambda) = \epsilon_{\mu_1\mu_2\mu_3\mu_4}\Lambda^{\mu_1}{}_0\Lambda^{\mu_2}{}_1\Lambda^{\mu_3}{}_2\Lambda^{\mu_4}{}_3,$$

from which, and the antysymmetric properties of $\epsilon_{\mu_1\mu_2\mu_3\mu_4}$, it follows.

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4}\Lambda^{\mu_1}{}_{\nu_1}\Lambda^{\mu_2}{}_{\nu_2}\Lambda^{\mu_3}{}_{\nu_3}\Lambda^{\mu_4}{}_{\nu_4} = \epsilon_{\nu_1\nu_2\nu_3\nu_4}\det(\Lambda).$$

Therefore, from the first expression for the determinant

$$S^{-1}\gamma^5 S = \frac{\mathrm{i}}{4!}\epsilon_{\nu_1\nu_2\nu_3\nu_4}\det(\Lambda)\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4} = \det(\Lambda)\gamma^5.$$

Substituting the last relation back into $\bar{\psi}'\gamma^\mu\gamma^5\psi'$ gives

$$\begin{aligned}\bar{\psi}'\gamma^\mu\gamma^5\psi' &= \Lambda^\mu{}_\nu\left(\det(\Lambda)\gamma^5\right)\psi \\ &= \det(\Lambda)\Lambda^\mu{}_\nu\left(\bar{\psi}\gamma^\nu\psi\right).\end{aligned}$$

The factor $\det(\Lambda)$ in the transformation law indicates that we are dealing with a **tensor density** quantity.

Comment: A tensor density transforms as a tensor when passing from one coordinate system to another, except that it is additionally multiplied or weighted by a power of the Jacobian determinant of the coordinate transition function or its absolute value. For example, a typical tensor density is the volume element where we add a scalar density $\sqrt{-\det g}$ to make the volume element transforming properly as a tensor, that is $\int \sqrt{-\det g}\mathrm{d}^4x$. Here we have a special case of a pseudotensor with sign-flip under an improper rotation (= a proper rotation followed by reflection).

In our case, the part $\Lambda^\mu{}_\nu\left(\bar{\psi}\gamma^\nu\psi\right)$ transforms as a vector, while $\det(\Lambda) = -1$ changes the sign in case of the spatial reflections, and consequently $\bar{\psi}'\gamma^\mu\gamma^5\psi'$ transforms as an **axial vector**.

To summarize, $\bar{\psi}\psi$ transforms as a scalar, $\bar{\psi}\gamma^\mu\psi$ as a vector and $\bar{\psi}\gamma^\mu\gamma^5\psi$ as an axial vector.

Problem 3, Massive vector bosons

Derive equations of motion for the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_W^2 W^\mu W_\mu$$

and interpret the results.

Proposed Solution

Let us look at the equation of motion of the massive vector boson. The equation of motion can be found from the Lagrangian

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_W^2 W^\mu W_\mu \\
&= -\frac{1}{4}\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\left(\partial^\mu W^\nu - \partial^\nu W^\mu\right) + \frac{1}{2}m_W^2 W^\mu W_\mu \\
&= -\frac{1}{2}\left(\partial_\mu W_\nu \partial^\mu W^\nu - \partial_\mu W_\nu \partial^\nu W^\mu - m_W^2 W^\mu W_\mu\right)
\end{aligned}$$

where $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu$. The Lagrangian density expands as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\left(\partial^\mu W^\nu - \partial^\nu W^\mu\right) + \frac{1}{2}m_W^2 W^\mu W_\mu \\
&= -\frac{1}{2}\left(\partial_\mu W_\nu \partial^\mu W^\nu - \partial_\mu W_\nu \partial^\nu W^\mu - m_W^2 W^\mu W_\mu\right) \\
&= -\frac{1}{2}\left(\partial_\nu W_\mu \partial^\nu W^\mu - \partial_\nu W_\mu \partial^\mu W^\nu - m_W^2 W^\mu W_\mu\right)
\end{aligned}$$

As usual, the equations of motion are obtained frome the Euler-Lagrange equation

$$0 = \frac{\partial \mathcal{L}}{\partial W_\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu W_\mu\right)}.$$

Evaulating the individual terms we get

$$\frac{\partial \mathcal{L}}{\partial W_\mu} = m_W^2 W^\mu$$

and

$$\begin{aligned}
\partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu W_\mu\right)} &= -\partial_\nu\left[\partial^\nu W^\mu - \partial^\mu W^\nu\right] \\
&= \partial^\mu \partial_\nu W^\nu - \partial_\nu \partial^\nu W^\mu \\
&= \partial^\mu \partial_\nu W^\nu - \Box W^\mu
\end{aligned}$$

thus combining we get the equations of motion (the **Proca** equation)

$$\boxed{\Box W^\mu + m_W^2 W^\mu - \partial^\mu \partial_\nu W^\nu = 0}.$$

Taking the divergence of the proca equation ∂_μ, we obtain

$$\begin{aligned} 0 &= \partial_\mu \left[\Box W^\mu + m_W^2 W^\mu - \partial^\mu \partial_\nu W^\nu\right] \\ &= \Box \partial_\mu W^\mu + m_W^2 \partial_\mu W^\mu - \partial_\mu \partial^\mu \partial_\nu W^\nu \\ &= \Box \partial_\mu W^\mu + m_W^2 \partial_\mu W^\mu - \Box \partial_\nu W^\nu \\ 0 &= m_W^2 \partial_\mu W^\mu \end{aligned}$$

or finally

$$\boxed{\partial_\mu W^\mu = 0}.$$

The last condition is a consequence of the Proca equation. This is very differently when compared to massless vector field (photons) where $\partial_\mu A^\mu = 0$ was a possible choice of fixing the gauge (the Lorenz gauge).

Thus for massive vector bosons, there is no freedom in choosing a guage.

In a sense, the Lorenz gauge has already been imposed by the equations of motion. This is a direct consequence on the number of linearly independent components of the fields W^μ.

Let us count the **degrees of freedom**:

- For massless vector boson A^μ there are $2 = 4 - 2$ independent components. One component can be removed by using the freedom in the choice of the gauge transformation and the other can be eliminated using the equations of motion and the *residual* gauge invariance.
- However, for a massive vector boson W^μ are actually $3 = 4 - 1$ independent components, where one component is removed by $\partial_\mu W^\mu = 0$ without any fixing the gauge.

Consequently, for $m_W \neq 0$ the Lagrangian is no longer invariant under gauge transformation

$$W_\mu \to W'_\mu = W_\mu + \mathrm{i}\partial_\mu \phi(x).$$

Indeed, $F_{\mu\nu} \to F'_{\mu\nu}$ is invariant, but

$$\begin{aligned} W^\mu W_\mu \to W'^\mu W'_\mu &= \left(W_\mu + \mathrm{i}\partial_\mu \phi(x)\right)\left(W^\mu + \mathrm{i}\partial^\mu \phi(x)\right) \\ W'^\mu W'_\mu &= W_\mu W^\mu + 2\mathrm{i}\partial_\mu \phi(x) W^\mu - \partial_\mu \phi(x) \partial^\mu \phi(x) \end{aligned}$$

is not invariant. Under *integration by parts*, it can be rewritten as

$$\begin{aligned} W^\mu W_\mu \to W'^\mu W'_\mu &= W_\mu W^\mu + 2\mathrm{i}\partial_\mu \phi(x) W^\mu - \partial_\mu \phi(x) \partial^\mu \phi(x) \\ &= W_\mu W^\mu + 2\mathrm{i}\partial_\mu [\underbrace{\phi(x) W^\mu}_{\text{compact support}}] - 2\mathrm{i}\phi(x)\partial_\mu W^\mu - \partial_\mu \phi(x) \partial^\mu \phi(x) \\ &= W_\mu W^\mu - 2\mathrm{i}\phi(x) \underbrace{\partial_\mu W^\mu}_{0 \text{ from EoM}} - \partial_\mu \phi(x) \partial^\mu \phi(x) \end{aligned}$$

$$= \quad W^2 - (\partial_\mu \phi)^2$$

but still, the contribution $-(\partial_\mu \phi)^2$ is out there. As the result, this theory is not normalizable.

How to deal with this? By using the Higgs mechanism. The early approach was the Stückelberg trick to add an extra scalar field. The Stückelberg action describes a massive spin-1 field as an Yang–Mills theory coupled to a real scalar field ϕ.

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2}(\partial^\mu \phi + mA^\mu)(\partial_\mu \phi + mA_\mu)$$

This is a special case of the Higgs mechanism, where, in effect, the mass of the Higgs scalar excitation has been taken to infinity, so the Higgs has decoupled and is ignorable. Gauge-fixing $\phi = 0$, yields the Proca action.

Problem 4, EoM for the Yang-Mills Lagrangian ☼

The Lagrangian density for SU(n) gauge fields (also called Yang-Mills Lagrangian) reads,

$$\mathcal{L}^{\mathrm{YM}} = -\frac{1}{4}\mathrm{Tr}F^2 == -\frac{1}{4}F^j_{\mu\nu}F^{\mu\nu j}.$$

Evaluate the equation of motion for A^i_μ, expressing it in covariant form.

Use the convention where the generators of the Lie algebra corresponding to the F-quantities are satisfying

$$[T_i, T_j] = \mathrm{i}f_{ijk}T_k, \qquad \mathrm{Tr}(T_iT_j) = \delta_{ij}. \tag{4.1}$$

Proposed Solution

Observe

$$\begin{aligned}
\mathcal{L}^{\mathrm{YM}} = -\frac{1}{4}\mathrm{Tr}F^2 &= -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}\mathrm{Tr}(F^j_{\mu\nu}T_j\,F^{\mu\nu k}T_k) \\
&= -\frac{1}{4}F^j_{\mu\nu}F^{\mu\nu k}\mathrm{Tr}(T_jT_k) = -\frac{1}{4}F^j_{\mu\nu}F^{\mu\nu k}\delta_{jk} = -\frac{1}{4}F^j_{\mu\nu}F^{\mu\nu j}.
\end{aligned}$$

We start from

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu]}$$

(The last relation can be derived by the commutator $[D_\mu, D_\nu] = \mathrm{i}gF_{\mu\nu} = \mathrm{i}gT^jF^j_{\mu\nu}$ where the covariant derivative is defined as$D_\mu = \partial_\mu + \mathrm{i}gT^jA^j_\mu$. Note: the sign in front of g depends on the convention used.)

After expanding into terms of the generators T_j, we get

$$\begin{aligned}
F^j_{\mu\nu}T_j &= \partial_\mu A^j_\nu T_j - \partial_\nu A^j_\mu T_j + \mathrm{i}g\left((A^j_\mu T_j)(A^k_\nu T_k) - (A^j_\nu T_j)(A^k_\mu T_k)\right) \\
&= \partial_\mu A^j_\nu T_j - \partial_\nu A^j_\mu T_j + \mathrm{i}g\left(A^j_\mu A^k_\nu - A^j_\nu A^k_\mu\right)T_jT_k \\
&= \partial_\mu A^j_\nu T_j - \partial_\nu A^j_\mu T_j + \mathrm{i}gA^j_\mu A^k_\nu[T_j, T_k] \\
&= \partial_\mu A^j_\nu T_j - \partial_\nu A^j_\mu T_j + \mathrm{i}gA^j_\mu A^k_\nu\left(\mathrm{i}f_{jkm}T_m\right), \\
\mathrm{Tr}(F^j_{\mu\nu}T_jT_i) &= \mathrm{Tr}\left(\partial_\mu A^j_\nu T_jT_i - \partial_\nu A^j_\mu T_jT_i - gf_{jkm}A^j_\mu A^k_\nu T_mT_i\right).
\end{aligned}$$

Now, from $\mathrm{Tr}(T_iT_j) = \delta_{ij}$,

$$\boxed{F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - gf^{ijk}A^j_\mu A^k_\nu}.$$

The equations of motion can be dervied from the Euler-Lagrange equations,

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^i_\nu)} - \frac{\partial\mathcal{L}}{\partial A^i_\nu} = 0. \tag{4.2}$$

We start with the evaluation of the second term, which gives,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A^i_\nu} &= \frac{\partial}{\partial A^i_\nu}\left(-\frac{1}{4}F^a_{\rho\sigma}F^{\rho\sigma a}\right) \\
&= \frac{\partial}{\partial A_\nu}\left(-\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{ajk}A^j_\rho A^k_\sigma\right)F^{\rho\sigma a}\right) \\
&= gf^{aji}A^j_\rho F^{\rho\nu a} = -gf^{iba}A^b_\mu F^{\mu\nu a} = -gf^{ijk}A^j_\mu F^{\mu\nu k}
\end{aligned}$$

The evaluation of the first term is slightly more complicated if done first time, but we may reuse what we know from the Maxwell field,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^i_\nu)} = \frac{\partial}{\partial(\partial_\mu A^i_\nu)}\left(-\frac{1}{4}F^a_{\rho\sigma}F^{\rho\sigma a}\right) = -F^{\mu\nu i}$$

Now substituting the last two expressions back into (4.2)

$$\partial_\mu(-F^{\mu\nu i}) - (-gf^{ijk}A^j_\mu F^{\mu\nu k}) = 0,$$

$$\partial_\mu F^{\mu\nu i} = gf^{ijk}A^j_\mu F^{\mu\nu k}.$$

Problem 5, SU(2) charges of weak interactions

When assigning charges to different object, it is not relevant how many different operators exist in a theory. Rather it is important how many operators can be found that simultaneously commutes. In this course, we are looking at weak interactions, with is a SU(2) theory. As such there are three generators τ_1, τ_2 and τ_3 with the corresponding charges Q_1, Q_2 and Q_3.

However, as these generators do not commute

$$[\tau_i, \tau_j] = \mathrm{i}\epsilon_{ijk}\tau_k,$$

and neither does the charges

$$[Q_i, Q_j] = \mathrm{i}\epsilon_{ijk}Q_k.$$

To get a feeling for how this works, we may review an equivalent, by the familiar problem.

The spin assignment in SU(2)

The particle spin is described by the SU(2) operators J_x, J_y and J_z depending on the axis of quantization. These operators also satisfy the dening relations for SU(2)

$$[\, J_i,\ J_j\,] = \mathrm{i}\epsilon_{ijk}J_k.$$

given by

$$J_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\sigma_x, \qquad J_x = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \frac{1}{2}\sigma_y, \qquad J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_z.$$

Here, a maximal set of commuting matrices are given by just one of the J's. This reflects the notion that spin is only well dened in one direction at a time. As is familiar it is possible to construct a linear combination

$$J_\pm = J_x \pm \mathrm{i}J_y$$

such that

$$[\, J_z,\ J_\pm\,] = \pm J_\pm.$$

In this sense $J_\pm$ are acting as the raising and lowering operators for the spin.

In a similar fashion it is possible to construct the charges

$$Q_\pm = Q_1 \pm \mathrm{i}Q_2$$

that has the commutation relation

$$[\,Q_3,\,Q_\pm\,] = \pm Q_\pm.$$

Thus in the same way as J change the spin, so does Q change the weak charge. This ties into the W boson in terms of the conserved currents

$$J_i^\mu = \frac{1}{2}\bar{\Psi}^{\mathrm{L}}\gamma^\mu\tau_i\Psi^{\mathrm{L}}$$

where

$$\Psi = \begin{pmatrix}\psi_{\nu_l}\\ \psi_l\end{pmatrix}, \qquad \Psi^{\mathrm{L}} = \begin{pmatrix}\psi_{\nu_l}^{\mathrm{L}}\\ \psi_l^{\mathrm{L}}\end{pmatrix}$$

and τ_i are the Pauli matrices. The three currents are then

$$\begin{aligned}
J_1^\mu &= \frac{1}{2}\left\{\bar{\psi}_{\nu_l}^{\mathrm{L}}\gamma^\mu\psi_l^{\mathrm{L}} + \bar{\psi}_l^{\mathrm{L}}\gamma^\mu\psi_{\nu_l}^{\mathrm{L}}\right\}\\
J_2^\mu &= \frac{1}{2}\left\{\bar{\psi}_{\nu_l}^{\mathrm{L}}\gamma^\mu\psi_l^{\mathrm{L}} - \bar{\psi}_{\nu_l}\gamma^\mu\psi_l^{\mathrm{L}}\right\}\\
J_3^\mu &= \frac{1}{2}\left\{\bar{\psi}_l^{\mathrm{L}}\gamma^\mu\psi_l^{\mathrm{L}} + \bar{\psi}_{\nu_l}\gamma^\mu\psi_{\nu_l}^{\mathrm{L}}\right\}
\end{aligned}$$

We can see that

$$J_3 = J_l^{\mathrm{L}} + J_{\nu_l}^{\mathrm{L}}$$

is a combination of the left-handed electron an neutrino currents.

The other two currents are tricker, by we can write them as

$$\begin{aligned}
J^\mu &= J_1^\mu + \mathrm{i}J_2^\mu = \bar{\psi}_{\nu_l}^{\mathrm{L}}\gamma^\mu\psi_l^{\mathrm{L}}\\
J^{\dagger\mu} &= J_1^\mu - \mathrm{i}J_2^\mu = \bar{\psi}_l^{\mathrm{L}}\gamma^\mu\bar{\psi}_{\nu_l}^{\mathrm{L}}
\end{aligned}$$

The point here is that $Q_3 = J_3^0$ and $Q_+ = J^0$ and $Q_- = J^{\dagger 0}$. Thus the currents J^μ and $J^{\dagger\mu}$ carry weak charges ± 1, which can be seen from $[\,Q_3,\,Q_\pm\,] = \pm Q_\pm$.