Quantum Field Theory
Path Integrals

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1 Path integrals - Mathematical Preliminaries

Based on Altland & Simons 3.2

In order to treat any path integrals we will need some mathematical preliminaries. The most important will be how to perform Gaussian integrals in higher dimensions. We start with the simple Gaussian integral in one dimension

$$I_{a,b} = \int_{-\infty}^{\infty} dv e^{-\frac{a}{2}v^2+bv} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}$$

by analytical continuation, this formula is valid also for complex a and b as long as $\Re(a) > 0$. From here it is straight forward to generalize to a matrix structure. Consider the integral

$$I_{\mathbf{A}} = \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}}$$

where the integration now is over the N-dimensional coordinate $\mathbf{v}$ and $\mathbf{A}$ is a positive definite real symmetric matrix. These matrices can always be diagonalized with an orthonormal coordinate transformation $\mathbf{O}$ such that $\mathbf{A} = \mathbf{O}^T\mathbf{D}\mathbf{O}$ and as $\mathbf{O}$ is orthonormal $\mathbf{v} \to \mathbf{O}^T\mathbf{v}$ with unit Jacobian. Thus

$$I_{\mathbf{A}} = \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{O}^T\mathbf{D}\mathbf{O}\mathbf{v}} = \left[\mathbf{v} \to \mathbf{O}^T\mathbf{v}\right] = \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{O}\mathbf{O}^T\mathbf{D}\mathbf{O}\mathbf{O}^T\mathbf{v}} = \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{D}\mathbf{v}}$$

and as $\mathbf{D}$ is diagonal we may split the integral as

$$\begin{aligned} I_{\mathbf{A}} &= \prod_{i=1}^{N}\left(\int_{-\infty}^{\infty} dv_i\, e^{-\frac{1}{2}D_i v_i^2}\right) = \prod_{i=1}^{N}\left(\sqrt{\frac{2\pi}{D_i}}\right) \\ &= \frac{\sqrt{2\pi}^N}{\sqrt{\det(\mathbf{D})}} = \frac{\sqrt{2\pi}^N}{\sqrt{\det(\mathbf{A})}} \end{aligned}$$

The last two steps happens as $\det\mathbf{D} = \prod_i D_i$ and $\det\mathbf{D} = \det\mathbf{A}$. From here is simple to also include a linear term $e^{\mathbf{j}^T\mathbf{v}}$ as

$$I_{\mathbf{A},\mathbf{j}} = \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\mathbf{v}} = \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\left(\mathbf{v}^T\mathbf{A}\mathbf{v}-\mathbf{j}^T\mathbf{v}-\mathbf{v}^T\mathbf{j}\right)}.$$

We proceed by completing the multidimensional square as $\left(\mathbf{v}^T - \mathbf{j}^T\mathbf{A}^{-1}\right)\mathbf{A}\left(\mathbf{v}-\mathbf{A}^{-1}\mathbf{j}\right) = \mathbf{v}^T\mathbf{A}\mathbf{v} - \mathbf{v}^T\mathbf{j} - \mathbf{j}^T\mathbf{v} + \mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}$ such that

$$\begin{aligned} I_{\mathbf{A},\mathbf{j}} &= \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\mathbf{v}} = e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\left(\mathbf{v}^T-\mathbf{j}^T\mathbf{A}^{-1}\right)\mathbf{A}\left(\mathbf{v}-\mathbf{A}^{-1}\mathbf{j}\right)} \\ &= [\text{shift } \mathbf{v}] = e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} \\ &= I_{\mathbf{A}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\mathbf{v}} = e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \frac{\sqrt{2\pi}^N}{\sqrt{\det\left(\mathbf{A}\right)}} \tag{1}$$

2 Wick's Theorem

The final identity above gives us a way to understand both mow the Feynman rules and Wick's theorem arise using path integrals. The trick is to view the linear term $\mathbf{j}^T\mathbf{v}$ as a source for $\mathbf{v}$ and use it to generate identities. For instance, we may take the derivative w.r.t. j_k and we get

$$\int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\mathbf{v}} v_k = \frac{\partial}{\partial j_k} I_{\mathbf{A},\mathbf{j}} = I_{\mathbf{A}} \frac{\partial}{\partial j_k} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} = I_{\mathbf{A}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \left(\mathbf{A}^{-1}j\right)_k$$

where we used the identity

$$\frac{\partial}{\partial j_k}\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j} = \frac{1}{2}\frac{\partial}{\partial j_k}\sum_{il}\left(j_i A^{-1}_{il} j_l\right) = \sum_l A^{-1}_{kl} j_l = \left(\mathbf{A}^{-1}j\right)_k$$

Differentiating once more we have

$$\frac{\partial}{\partial j_l}\left(\mathbf{A}^{-1}j\right)_k = A^{-1}_{kl}$$

such that

$$\begin{aligned} \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}+\mathbf{j}^T\mathbf{v}} v_l v_k &= \frac{\partial}{\partial j_k}\frac{\partial}{\partial j_l} I_{\mathbf{A},\mathbf{j}} = I_{\mathbf{A}} \frac{\partial}{\partial j_l}\frac{\partial}{\partial j_k} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \\ &= I_{\mathbf{A}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \left(\mathbf{A}^{-1}j\right)_l \left(\mathbf{A}^{-1}j\right)_k + I_{\mathbf{A}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} A^{-1}_{lk} \end{aligned}$$

This may look messy, but putting $\mathbf{j} = 0$ at the end of the calculation leads to

$$\int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} v_l v_k = I_{\mathbf{A}} A^{-1}_{lk}$$

Introducing the notation $\langle \dots \rangle = \frac{1}{I_{\mathbf{A}}} \int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v}} \left(\dots\right)$ we have just proved

$$\langle v_k v_l \rangle = A^{-1}_{lk} \tag{2}$$

Thus we started at

$$\left\langle e^{\mathbf{j}^T\mathbf{v}} \right\rangle = e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \tag{3}$$

and by taking two derivatives we arrived at 2. Taking two more derivatives we have

$$
\begin{aligned}
& \partial_{j_{m_1}}\partial_{j_{m_2}}\partial_{j_{m_3}}\partial_{j_{m_4}}\left\langle e^{\mathbf{j}^T\mathbf{v}}\right\rangle = \left\langle e^{\mathbf{j}^T\mathbf{v}} v_{m_1}v_{m_2}v_{m_3}v_{m_4}\right\rangle \\
& \| \\
& \partial_{j_{m_1}}\partial_{j_{m_2}}\partial_{j_{m_3}}\partial_{j_{m_4}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}} \\
= \; & \partial_{j_{m_2}}\partial_{j_{m_3}}\partial_{j_{m_4}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\left(\mathbf{A}^{-1}j\right)_{m_1} \\
= \; & \partial_{j_{m_3}}\partial_{j_{m_4}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\left[\left(\mathbf{A}^{-1}j\right)_{m_2}\left(\mathbf{A}^{-1}j\right)_{m_1} + A^{-1}_{m_1m_2}\right] \\
= \; & \partial_{j_{m_4}} e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\Big[\left(\mathbf{A}^{-1}j\right)_{m_3}\left(\mathbf{A}^{-1}j\right)_{m_2}\left(\mathbf{A}^{-1}j\right)_{m_1} + A^{-1}_{m_1m_3}\left(\mathbf{A}^{-1}j\right)_{m_2} + \\
& \quad + A^{-1}_{m_2m_3}\left(\mathbf{A}^{-1}j\right)_{m_1} + A^{-1}_{m_1m_2}\left(\mathbf{A}^{-1}j\right)_{m_3}\Big] \qquad (4) \\
= \; & e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\left[\left(\mathbf{A}^{-1}j\right)_{m_3}\left(\mathbf{A}^{-1}j\right)_{m_2}\left(\mathbf{A}^{-1}j\right)_{m_1}\left(\mathbf{A}^{-1}j\right)_{m_4}\right] \\
& + e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\Big[A^{-1}_{m_1m_2}\left(\mathbf{A}^{-1}j\right)_{m_3}\left(\mathbf{A}^{-1}j\right)_{m_4} + A^{-1}_{m_1m_3}\left(\mathbf{A}^{-1}j\right)_{m_2}\left(\mathbf{A}^{-1}j\right)_{m_4} \qquad (5) \\
& \qquad + A^{-1}_{m_1m_4}\left(\mathbf{A}^{-1}j\right)_{m_3}\left(\mathbf{A}^{-1}j\right)_{m_2} + \qquad (6) \\
& \qquad + A^{-1}_{m_2m_3}\left(\mathbf{A}^{-1}j\right)_{m_1}\left(\mathbf{A}^{-1}j\right)_{m_4} + A^{-1}_{m_2m_4}\left(\mathbf{A}^{-1}j\right)_{m_3}\left(\mathbf{A}^{-1}j\right)_{m_1} + \\
& \qquad + A^{-1}_{m_3m_4}\left(\mathbf{A}^{-1}j\right)_{m_2}\left(\mathbf{A}^{-1}j\right)_{m_1}\Big] \qquad (7) \\
& + e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\left[A^{-1}_{m_1m_2}A^{-1}_{m_3m_4} + A^{-1}_{m_1m_3}A^{-1}_{m_2m_4} + A^{-1}_{m_2m_3}A^{-1}_{m_1m_4}\right]
\end{aligned}
$$

Finally setting $\mathbf{j} = 0$ gives

$$\langle v_{m_1}v_{m_2}v_{m_3}v_{m_4}\rangle = A^{-1}_{m_1m_3}A^{-1}_{m_2m_4} + A^{-1}_{m_2m_3}A^{-1}_{m_1m_4} + A^{-1}_{m_1m_2}A^{-1}_{m_3m_4} \qquad (8)$$

As should be clear from the above example, going to higher and even number of derivative will give the result

$$\left\langle \prod_{i=1}^{2n} v_{m_i}\right\rangle = \sum_{k_j\in\{\text{all paring of } m_i\}} A^{-1}_{m_{k_1}m_{k_2}}\dots A^{-1}_{m_{k_{2n-1}}m_{k_{2n}}} \qquad (9)$$

whereas to such identity exits for an odd number of insertions of v.

From here it should be clear why Wick's theorem look the way it does. Looking at the last piece of 5 we see that we may identify all the pieces to Wick's theorem, including 0-contractions, 1-contractions and 2-contractions.

2.1 The propagator and the equation of motion

From (2) it is suggestive why the propagator is the inverse of the equation of motion. Assume for instance that A_{ik} is the Fourier version of the scalar field Lagrangian $\partial_\mu\partial^\mu - m^2$ which is $A_{p_ip_j} = \mathcal{L}_{p_ip_j} = \delta_{p_ip_j}\left(p_{i,\mu}p_i^\mu - m^2\right)$. The the two-point function is then

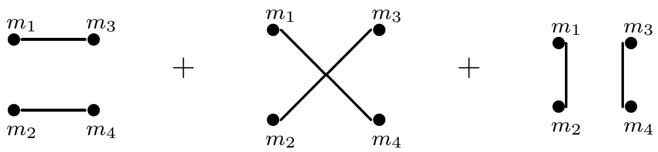
$$\left\langle v_{p_i}v_{p_j}\right\rangle = A^{-1}_{p_ip_j} = \frac{\delta_{p_ip_j}}{p_i^2 - m^2}$$

which is exactly the propagator.

2.2 Feynman diagram structures

To keep the notation down, you will take the first steps towards Feynman rules and diagrams without leaving the regularized N-dimensional space. We now think more specifically of A^{-1} as the greens function or propagator of a non-interacting system. For instance the thee terms in 2 and 8 may graphically be depicted by all the ways of going from the l to k

and the points m_1 and m_2 to m_3 and m_4 as



(In a system with causality the last one will of course disappear).

We will now investigate the consequence of introducing an interaction into the system. The interaction will be of the the form $g\sum_i v_i^4$ such that there is an extra factor of $e^{-g\sum_i v_i^4}$ in the expectation value. Roughly speaking e are interested in evaluating the correlator $\langle v_{m_1} v_{m_2}\rangle$ in the presence of $e^{-g\sum_i v_i^4}$. The correlator is thus

$$\langle v_{m_1} v_{m_2}\rangle_{\text{Int}} = \left\langle v_{m_1} v_{m_2} e^{-g\sum_i v_i^4}\right\rangle$$

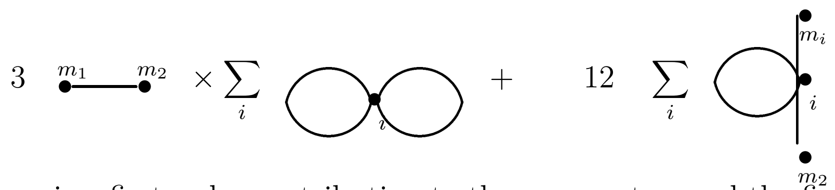
We precede by looking a the first order expansion of the exponential as

$$\langle v_{m_1} v_{m_2}\rangle_{\text{Int}} = \left\langle v_{m_1} v_{m_2}\left(1 - g\sum_i v_i^4 + \ldots\right)\right\rangle = \langle v_{m_1} v_{m_2}\rangle - g\left\langle v_{m_1} v_{m_2}\sum_i v_i^4\right\rangle + \ldots$$

The first piece is the ordinary propagator, but the second price contains the first non-trivial diagrams. Using the wick expansion rules from (9) we have

$$\begin{aligned}\left\langle v_{m_1} v_{m_2}\sum_i v_i^4\right\rangle &= A^{-1}_{m_1 m_2}\sum_i\left\langle v_i^4\right\rangle + 4\sum_i A^{-1}_{m_1 i}\left\langle v_{m_2} v_i^3\right\rangle \\ &= 3A^{-1}_{m_1 m_2}\sum_i\left(A^{-1}_{ii}\right)^2 + 12\sum_i A^{-1}_{m_1 i}A^{-1}_{m_2 i}A^{-1}_{ii}\end{aligned}$$

These two terms correspond to the diagrams



The second diagram is a first order contribution to the propagator, and the first term is a bare propagator and a vacuum bubble.

3 Path integrals - Effective theories

Based on Altland & Simons 3.2

Lets take a look at an interacting theory, not unlike QED, but well stock with real bosons for simplicity. The theory is

$$\begin{aligned}\mathcal{L}_\phi &= \frac{1}{2}\phi\left(\partial^2 - m^2\right)\phi \\ \mathcal{L}_\varphi &= \frac{1}{2}\varphi\left(\partial^2 - M^2\right)\varphi \end{aligned} \tag{10}$$

which are the ordinary non-interacting pieces. We also have an interaction

$$\mathcal{L}_I = \imath g\phi^2(x)\,\varphi(x)$$

Where the existence of the $\imath$ will be apparent soon. Here the field φ plays a role similar to that of a photon field. We would not like to know what the effective theory is at low momentum. For this purpose we may use the path integrals. We will still work in regularized version to keep the arithmetics simple - but the structure is the same in the full-fledged version also.

The path integral correlator will now we written

$$\langle \dots \rangle = \int_{-\infty}^{\infty} d^N\phi \int_{-\infty}^{\infty} d^N\varphi\, e^{-S_{\phi,\varphi}}(\dots)$$

where

$$S_{\phi,\varphi} = \frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j + \frac{1}{2}\sum_{i,j}\varphi_i B_{ij}\varphi_j + \imath g\sum_i \phi_i^2\varphi_i.$$

Here A_{ij} and B_{ij} are the regularized versions of (10). As the interaction is written the path integral is in real space.

We not imagine that $m \ll M$ such that free φ-particles are rarely or never seen, except at high energies. In those cases the only relevant correlators would be of the type $\langle\prod_i^n \phi_i\rangle$ such that there are no φ contributions.[1] We may then write

$$\left\langle \prod_i^n \phi_{k_i}\right\rangle = \int_{-\infty}^{\infty} d^N\phi \int_{-\infty}^{\infty} d^N\varphi \exp\left\{-\frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j - \frac{1}{2}\sum_{i,j}\varphi_i B_{ij}\varphi_j - \imath g\sum_i \phi_i^2\varphi_i\right\}\prod_i^n \phi_{k_i}$$

and integrate out the φ contribution. In doing so we close our eyes for the existence φ-field but instead obtain an effective interaction for the ϕ-fields (which will be ϕ^4 at low momentum). For this we use the relation (given in tutorial 15)

$$\int_{-\infty}^{\infty} d\mathbf{v}\, e^{-\frac{1}{2}\mathbf{v}^T\mathbf{A}\mathbf{v} + \mathbf{j}^T\mathbf{v}} = e^{\frac{1}{2}\mathbf{j}^T\mathbf{A}^{-1}\mathbf{j}}\frac{\sqrt{2\pi}^N}{\sqrt{\det(\mathbf{A})}} \tag{11}$$

Using this relation on the field φ gives

$$\left\langle \prod_i^n \phi_{k_i}\right\rangle = \frac{\sqrt{2\pi}^N}{\sqrt{\det(\mathbf{B})}}\int_{-\infty}^{\infty} d^N\phi \exp\left\{-\frac{1}{2}\sum_{i,j}\phi_i A_{ij}\phi_j\right\} e^{\frac{1}{2}\sum_{i,j}\left(-\imath g\phi_i^2\right)B_{ij}^{-1}\left(-\imath g\phi_j^2\right)}\prod_i^n \phi_{k_i}$$

Effectively we have now introduced a new interaction on the form

$$\mathcal{L}_{II} = -\frac{1}{2}g^2\sum_{i,j}\phi_i^2 B_{ij}^{-1}\phi_j^2.$$

As it stands this effective theory is non-local as it contains a factor $\phi^2(x)\,B^{-1}(x,y)\,\phi^2(y)$. We will hover see that at large distance $B^{-1}(x,y) \approx \delta(x-y)$.

[1] Of course there exists correaltors that contain one or more final φ-particles, but these will be very unlikely.

3.1 Momentum

Do do so we transfer to momentum space where the original interaction would be written

$$\mathcal{L}_I = \sum_k \overbrace{\imath g \sum_{p_1,p_2} \delta_{p_1+p_2,k} \tilde{\phi}_{p_1} \tilde{\phi}_{p_2}}^{J_k} \tilde{\varphi}_k$$

as the interaction was translation invariant. Likewise the propagator would have been

$$\tilde{B}^{-1}_{k,k'} = \frac{\delta_{k,k'}}{k^2 + M^2}$$

Using the result (11) would give the effective interaction

$$\begin{aligned} \mathcal{L}_{II} &= \frac{1}{2} \sum_{k,k'} J_k \tilde{B}^{-1}_{k,k'} J_{k'} \\ &= \frac{1}{2} \sum_{k,k'} \left(\imath g \sum_{p_1,p_2} \delta_{p_1+p_2,k} \tilde{\phi}_{p_1} \tilde{\phi}_{p_2} \right) \frac{\delta_{k,k'}}{k^2 + M^2} \left(\imath g \sum_{p_1',p_2'} \delta_{p_1'+p_2',k'} \tilde{\phi}_{p_1'} \tilde{\phi}_{p_2'} \right) \\ &= -\frac{1}{2} g^2 \sum_k \sum_{p_1,p_2} \sum_{p_1',p_2'} \tilde{\phi}_{p_1} \tilde{\phi}_{p_2} \frac{\delta_{p_1+p_2,k} \delta_{p_1'+p_2',k}}{k^2 + M^2} \tilde{\phi}_{p_1'} \tilde{\phi}_{p_2'} \\ &= -\frac{1}{2} g^2 \sum_{p_1,p_2} \sum_{p_1',p_2'} \tilde{\phi}_{p_1} \tilde{\phi}_{p_2} \frac{\delta_{p_1+p_2,p_1'+p_2'}}{q^2 + M^2} \tilde{\phi}_{p_1'} \tilde{\phi}_{p_2'} \end{aligned}$$

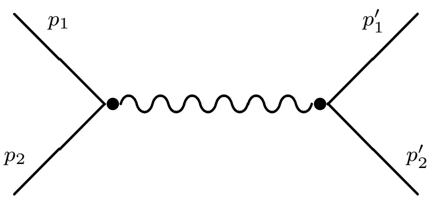
where $q = p_1 + p_2$. As you can see this is the structure we are already familiar with from propagators and operators formalism. Now wee look at low momentum such that $(p_1 + p_2)^2 \approx 4m^2 \ll M^2$. At low momentum the q^2 vanished compared to the M^2piece and we have

$$\mathcal{L}_{II} = -\frac{1}{2} \frac{g^2}{M^2} \sum_{p_1,p_2} \sum_{p_1',p_2'} \tilde{\phi}_{p_1} \tilde{\phi}_{p_2} \tilde{\phi}_{p_1'} \tilde{\phi}_{p_2'} \delta_{p_1+p_2,p_1'+p_2'}.$$

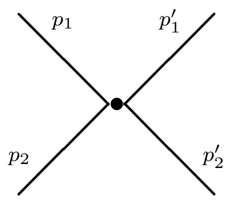
Fourier transforming back to real space this precisely gives the interaction

$$\mathcal{L}_{II} = -\frac{1}{2} \frac{g^2}{M^2} \sum_i \phi_i^4$$

which is an example of how ϕ^4 theory may arise out of an underlying $\phi^2 \varphi$.
That this should happen should not be to surprising as the diagram



at low momenta (long distances), becomes



such that the mediating interaction is difficult to see. The real space picture of the above statements is that a massive force carrier has a potential on the form

$$V(r) = \frac{e^{-Mr}}{r},$$

which investigated at $r \gtrsim \frac{1}{M}$ looks like $V(r) \approx \delta(r)$.

4 Fermions and Grassmann Numbers

See Mandl & Shaw 12.4.2, 13.1.2 & 13.1.3, or Altland & Simons 4.1.2

When we construct path integrals for fermions we naturally run in to a problem. For bosons, we replaced the quantum fields with scalar of vector numbers $\hat{\phi} \to \phi$ and $\hat{A}^\mu \to A^\mu$, but what should be do with the fermions?

We know that two fermions operators anti-commute as $\{\psi_i, \psi_j\} = 0$, this means that the numbers that we replace them with must also anti-commute. Thus, we make the replacement $\psi \to \eta$ where

$$\eta_i \eta_j = -\eta_j \eta_i \tag{12}$$

These objects are scalar, but not numbers in the ordinary sense. At best, we may think of them as some kind of matrices....

The algebra of these numbers work as follows. We may define a derivative operators with respect to η_i that is $\partial_i = \frac{\partial}{\partial \eta_i}$ such that

$$\partial_i \eta_j = \delta_{ij} \tag{13}$$

in analogy with ordinary c-numbers. However since

$$\partial_i \eta_j \eta_i = -\partial_i \eta_i \eta_j = -\eta_j$$

then for consistency

$$\partial_i \eta_j \eta_i = -\eta_j \partial_i \eta_i = -\eta_j$$

such that $\partial_i \eta_j = -\eta_j \partial_i + \delta_{ij}$. Thus Grassmann variables η and their derivatives $\frac{\partial}{\partial \eta}$ also anti-commute.

Grassmans may be multiplied by any complex number and thus forms a vector space

$$c_0 + c_i \eta_i + c_j \eta_j \in \mathcal{A}$$

For closure two objects $\alpha, \beta \in \mathcal{A}$ forms an object $\alpha\beta \in \mathcal{A}$. As $\eta_i^2 = 0$ any object in $\mathcal{A}$ may be written

$$\alpha = c_0 + \sum_i c_i \eta_i + \sum_{i<j} c_{ij} \eta_i \eta_j + \sum_{i<j<k} c_{ijk} \eta_i \eta_j \eta_k + \cdots \in \mathcal{A}$$

This is also the way to define functions of Grassmann variables. A function of only one Grassmann variable $f(\eta)$ may be defined by it's Taylor expansion:

$$f(\eta) = f(0) + \eta \left. \frac{\partial}{\partial \eta} f(\eta) \right|_{\eta=0}$$

then it stops as $\eta^2 = 0$. Similarly for multi-variable functions

$$f(\{\eta_i\}) = f(0) + \sum_i \eta_i \, \partial_i f|_{\{\eta\}=0} + \sum_{i<j} \eta_i \eta_j \, \partial_i \partial_j f|_{\{\eta\}=0} + \sum_{i<j<k} \eta_i \eta_j \eta_k \, \partial_i \partial_j \partial_k f|_{\{\eta\}=0} + \ldots$$

From this we ca see that many fermionic functions are actually the same, as there are only tow components in the Taylor-Expansion. For instance $e^{\alpha\eta} = \left(1 + \frac{\alpha\eta}{2}\right)^2$ since

$$\left(1 + \frac{\alpha\eta}{2}\right)^2 = 1 + 2\frac{\alpha\eta}{2} + \frac{\alpha^2\eta^2}{4} = 1 + \alpha\eta = e^{\alpha\eta}$$

We may also define a Grassmann integral as

$$\int d\eta\, 1 = 0 \tag{14}$$

$$\int d\eta\, \eta = 1 \tag{15}$$

This does look funny. But it is apparently the only reasonable definition. Using these definitions we may compute the Grassmann Gaussian integral as

$$\int d\eta\, d\bar{\eta}\, e^{\alpha\bar{\eta}\eta} = \int d\eta\, d\bar{\eta}\, (1 + \alpha\bar{\eta}\eta) = \int d\eta\, \alpha\eta = \alpha$$

Also for a diagonal matrix $\alpha_{ij} = \delta_{ij}\alpha_i$:

$$\int \prod_{i=1}^{N} (d\eta_i\, d\bar{\eta}_i)\; e^{\sum_{i=1}^{1} \bar{\eta}_i \alpha_i \eta_i} = \prod_{i=1}^{N} \left(\int d\eta_i\, d\bar{\eta}_i\, e^{\bar{\eta}_i \alpha_i \eta_i} \right) = \prod_{i=1}^{N} \alpha_i = \det \alpha$$

If α is a hermitian matrix it it possible to diagonalize it using unitary transformations. As such the more generic result

$$\int \prod_{i=1}^{N} (d\eta_i\, d\bar{\eta}_i)\; e^{\sum_{i=1}^{1} \bar{\eta}_i \alpha_{ij} \eta_i} = \det \alpha$$

also holds. Notice here that η and $\bar{\eta}$ are two distinct variables, otherwise the integral would vanish.