Tutorial 4

Topics for today's tutorial (in continuation of Tutorial 3)

The last time: **Part I**. The inverse operator expansion, a and $a^\dagger$ as functions of ϕ and $\dot{\phi}$. The evaluation of the commutator $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]$. The evaluation of the equal time commutator $[\phi(x), \phi(y)]_{x^0=y^0=t})$.

Now, **Part II**. The expansion of the Hamiltonian. An introduction to the infinite contributions to the energy and to the normal ordering of operators.

Next tutorial (as a part of Symmetries and Conservation Laws and Noether's theorem): Group theory. Representation of a group. Group Parameters. Lie Groups. The Rotation Group, SO(N); Unitary Groups, SU(N).

Problems

For the real scalar field $\phi(x)$: (1) Evaluate the commutator $[\phi(x), \pi(x')]$ at ET. (2) Evaluate $\partial_t\phi(x)$ and $\boldsymbol{\nabla}\phi(x)$. (3) Expand the Hamiltonian in terms of the creation and annihilation operators.

The canonical quantization procedure (reminder)

- FIRSTLY, (1) State the field and state the Lagrangian density. (2) Vary the action and derive the equations of motion. (3) Solve the equations of motion. (This is not always possible!) (4) In order to quantize this classical theory by the canonical formalism, we must introduce conjugate variables; thus go to Hamiltonian formalism and find $\phi(x)$ and $\pi(x)$. (5) Promote $\phi(x)$ and $\pi(x)$ to operators and impose either the equal-time commutation relations (ETCR) for integer-spin fields, or the equal-time anticommutation relations (ETaCR) for spinor fields. Once we imposed ETCR/ETaCR, we get the quantum theory. The field operators will be expressed in terms of creation and annihilation operators (which will form a polynomial Heisenberg algebra for the integer-spin fields).

- THEN, from the symmetries of the action, find the conserved quantities ('charges' or 'constants of motion'), e.g., the *canonical Hamiltonian density* $\mathcal{H}$ and the Hamiltonian H for the scalar field are

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\dot{\phi}^2(x) + (\boldsymbol{\nabla}\phi(x))\cdot(\boldsymbol{\nabla}\phi(x)) + m^2\phi^2(x)\right],$$

$$H = \int \mathrm{d}^3x\,\mathcal{H}(\phi,\pi,\partial_i\phi) = \int \mathrm{d}^3x\left(\dot{\phi}_a\pi^a - \mathcal{L}\right) = \frac{1}{2}\int \mathrm{d}^3x\,\left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi) + m^2\phi^2\right]$$

- To obtain the physical properties of particles, we express the constants of motion in terms of the creation and annihilation operators. For the Hamiltonian we get

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right)$$

- We need to handle the infinite expectation values for *operators* in the vacuum state. (Observe the infinite sum $\sum_{\mathbf{k}}\frac{1}{2}$ above.) Before quantization we had any order of *functions* in equations. This is resolved by prescribing the **normal ordering** of operators. (This topic will be covered in later tutorials in more detail!) The normal ordering is usually done at the level of Lagrangian assuming the operators from beginning. **N.B.** In Mandl and Shaw, from pedagogical reasons, the normal ordering is not done at the level of Lagrangian for a real KGF! Compare M&S Eqs. (3.4) and (3.22).

Symmetries and Conservation Laws & Noether's theorem (reminder)

Consider an action

$$S = \int \mathrm{d}^4x\, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{0.1}$$

where $\mathcal{L}$ is the Lagrangian density. From $\delta S = 0$, we get the *Euler-Lagrange equations of motion*

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{0.2}$$

The *cannonical momentum* conjugate to the field variable $\phi_a(x)$ is

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{0.3}$$

Noether's theorem
states that if the action is invariant with respect to the continous infinitesimal transformations[1]

$$x_\mu \to x'_\mu = x_\mu + \delta_\epsilon x_\mu, \qquad \delta_\epsilon x^\mu = x'^\mu - x^\mu, \tag{0.4}$$

$$\phi_a(x) \to \phi'_a(x') = \phi_a(x) + \delta_\epsilon \phi_a(x), \qquad \delta_\epsilon \phi_a(x) = \phi'_a(x') - \phi_a(x), \tag{0.5}$$

$$\bar{\delta}_\epsilon \phi_a(x) = \phi'_a(x) - \phi_a(x), \tag{0.6}$$

then the divergence of the *Noether current* J^μ_ϵ is equal to zero $\partial_\mu J^\mu_\epsilon = 0$,

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L} \delta_\epsilon x^\mu, \qquad \boxed{(!)\ J^\mu_\epsilon = \sum_{r=1}^{d} \epsilon_r J^\mu_{\epsilon_r}, \quad \epsilon = \{\epsilon_r\},} \tag{0.7}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_\epsilon \phi_a - T^\mu{}_\nu \delta_\epsilon x^\nu, \tag{0.8}$$

where $T^\mu{}_\nu$ is the *energy-momentum tensor*

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}\,. \tag{0.9}$$

The *Noether charges* Q_ϵ are constants of motion given by

$$Q_\epsilon = \int \mathrm{d}^3x\, J^0_\epsilon, \qquad Q_{\epsilon_r} = \int \mathrm{d}^3x\, J^0_{\epsilon_r}. \tag{0.10}$$

[1] The index ϵ is related to a symmetry group.
Note that $\bar{\delta}_\epsilon \phi_a(x)$ is calculated with both ϕ' and ϕ having the same argument x.