Tutorial 4

Topics for today's tutorial (in continuation of Tutorial 3)

Part I. (earlier) The inverse operator expansion, a and $a^\dagger$ as functions of ϕ and $\dot{\phi}$. The evaluation of the commutator $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]$. The evaluation of the equal time commutator $[\phi(x), \phi(y)]_{x^0=y^0=t})$.

Part II. (in these notes) The expansion of the Hamiltonian. An introduction to the infinite contributions to the energy and to the normal ordering of operators.

Problems

For the real scalar field $\phi(x)$: (1) Evaluate the commutator $[\phi(x), \pi(x')]$ at ET. (2) Evaluate $\partial_t\phi(x)$ and $\boldsymbol{\nabla}\phi(x)$. (3) Expand the Hamiltonian in terms of the creation and annihilation operators.

The canonical quantization procedure (reminder)

- FIRSTLY, (1) State the field and state the Lagrangian density. (2) Vary the action and derive the equations of motion. (3) Solve the equations of motion. (This is not always possible!) (4) In order to quantize this classical theory by the canonical formalism, we must introduce conjugate variables; thus go to Hamiltonian formalism and find $\phi(x)$ and $\pi(x)$. (5) Promote $\phi(x)$ and $\pi(x)$ to operators and impose either the equal-time commutation relations (ETCR) for integer-spin fields, or the equal-time anticommutation relations (ETaCR) for spinor fields. Once we imposed ETCR/ETaCR, we get the quantum theory. The field operators will be expressed in terms of creation and annihilation operators (which will form a polynomial Heisenberg algebra for the integer-spin fields).

- THEN, from the symmetries of the action, find the conserved quantities ('charges' or 'constants of motion'), e.g., the *canonical Hamiltonian density* $\mathcal{H}$ and the Hamiltonian H for the scalar field are

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left[\dot{\phi}^2(x) + (\boldsymbol{\nabla}\phi(x))\cdot(\boldsymbol{\nabla}\phi(x)) + m^2\phi^2(x)\right],$$

$$H = \int \mathrm{d}^3x\, \mathcal{H}(\phi,\pi,\partial_i\phi) = \int \mathrm{d}^3x \left(\dot{\phi}_a\pi^a - \mathcal{L}\right) = \frac{1}{2}\int \mathrm{d}^3x\, \left[\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\phi) + m^2\phi^2\right]$$

- To obtain the physical properties of particles, we express the constants of motion in terms of the creation and annihilation operators. For the Hamiltonian we get

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(a^\dagger(\mathbf{k})a(\mathbf{k}) + \frac{1}{2}\right)$$

- We need to handle the infinite expectation values for *operators* in the vacuum state. (Observe the infinite sum $\sum_{\mathbf{k}}\frac{1}{2}$ above.) Before quantization we had any order of *functions* in equations. This is resolved by prescribing the **normal ordering** of operators. (This topic will be covered in later tutorials in more detail!) The normal ordering is usually done at the level of Lagrangian assuming the operators from beginning. **N.B.** In Mandl and Shaw, from pedagogical reasons, the normal ordering is not done at the level of Lagrangian for a real KGF! Compare M&S Eqs. (3.4) and (3.22).

Symmetries and Conservation Laws & Noether's theorem (reminder)

Consider an action

$$S = \int \mathrm{d}^4x \, \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{0.1}$$

where $\mathcal{L}$ is the Lagrangian density. From $\delta S = 0$, we get the *Euler-Lagrange equations of motion*

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0. \tag{0.2}$$

The *cannonical momentum* conjugate to the field variable $\phi_a(x)$ is

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{0.3}$$

Noether's theorem
states that if the action is invariant with respect to the continous infinitesimal transformations[1]

$$x_\mu \to x'_\mu = x_\mu + \delta_\epsilon x_\mu, \qquad \delta_\epsilon x^\mu = x'^\mu - x^\mu, \tag{0.4}$$

$$\phi_a(x) \to \phi'_a(x') = \phi_a(x) + \delta_\epsilon \phi_a(x), \qquad \delta_\epsilon \phi_a(x) = \phi'_a(x') - \phi_a(x), \tag{0.5}$$

$$\bar{\delta}_\epsilon \phi_a(x) = \phi'_a(x) - \phi_a(x), \tag{0.6}$$

then the divergence of the *Noether current* J^μ_ϵ is equal to zero $\partial_\mu J^\mu_\epsilon = 0$,

$$J^\mu_\epsilon = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \bar{\delta}_\epsilon \phi_a + \mathcal{L} \delta_\epsilon x^\mu, \qquad \boxed{(!)\ J^\mu_\epsilon = \sum_{r=1}^{d} \epsilon_r J^\mu_{\epsilon_r}, \quad \epsilon = \{\epsilon_r\},} \tag{0.7}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta_\epsilon \phi_a - T^\mu{}_\nu \delta_\epsilon x^\nu, \tag{0.8}$$

where $T^\mu{}_\nu$ is the *energy-momentum tensor*

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L}\,. \tag{0.9}$$

The *Noether charges* Q_ϵ are constants of motion given by

$$Q_\epsilon = \int \mathrm{d}^3x \, J^0_\epsilon, \qquad Q_{\epsilon_r} = \int \mathrm{d}^3x \, J^0_{\epsilon_r}. \tag{0.10}$$

[1] The index ϵ is related to a symmetry group.
Note that $\bar{\delta}_\epsilon \phi_a(x)$ is calculated with both ϕ' and ϕ having the same argument x.

1 A convinient representation

To simplify further derivations, espcially those done later in the expansion of the Hamiltonian, let us introduce,

$$A_{\mathbf{k}}(x) = \frac{\mathrm{e}^{-ikx}}{\sqrt{2\omega_{\mathbf{k}}}} a_{\mathbf{k}}(x), \quad A^\dagger_{\mathbf{k}}(x) = \frac{\mathrm{e}^{ikx}}{\sqrt{2\omega_{\mathbf{k}}}} a^\dagger_{\mathbf{k}}(x)\,. \tag{1.1}$$

Using $kx = \omega_{\mathbf{k}} t - \mathbf{k}\cdot\mathbf{x}$, we can express and simplify various terms in the Hamiltonian as follows,

$$\begin{aligned}
\phi(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(A_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}(x)\right), \\
\dot\phi(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \frac{\partial}{\partial t} \left(A(x) + A^\dagger_{\mathbf{k}}(x)\right) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(-\mathrm{i}\omega_{\mathbf{k}} A_{\mathbf{k}}(x) + \mathrm{i}\omega_{\mathbf{k}} A^\dagger_{\mathbf{k}}(x)\right) \\
&= -\mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \omega_{\mathbf{k}} \left(A_{\mathbf{k}}(x) - A^\dagger_{\mathbf{k}}(x)\right), \\
\boldsymbol{\nabla}\phi(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \boldsymbol{\nabla} \left(A_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}(x)\right) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(\mathrm{i}\mathbf{k}\, A_{\mathbf{k}}(x) - \mathrm{i}\mathbf{k}\, A^\dagger_{\mathbf{k}}(x)\right) \\
&= \mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \mathbf{k} \left(A_{\mathbf{k}}(x) - A^\dagger_{\mathbf{k}}(x)\right).
\end{aligned}$$

2 The commutator $[\phi(x), \pi(x')]$

Yet again, let $x = (t, \mathbf{x})$ and $x' = (t, \mathbf{y})$. We shall compute the commutator at equal-time $x^0 = x'^0 = t$.

Firstly, in terms of $A_{\mathbf{k}}(x)$ and $A^\dagger_{\mathbf{k}}(x)$ from Eq. (1.1), we can simply obtain,

$$[\, A_{\mathbf{k}}(x),\, A^\dagger_{\mathbf{k}'}(x')\,] = \delta^3(\mathbf{k} - \mathbf{k}')\, \frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{2\omega_{\mathbf{k}}}.$$

Then we can evaluate,

$$\begin{aligned}
[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] &= \left[\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(A_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}(x)\right), -\mathrm{i} \int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}} \omega_{\mathbf{k}'} \left(A_{\mathbf{k}'}(x') - A^\dagger_{\mathbf{k}'}(x')\right)\right] \\
&= -\mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\, \omega_{\mathbf{k}'} \left[\left(A_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}(x)\right), \left(A_{\mathbf{k}'}(x') - A^\dagger_{\mathbf{k}'}(x')\right)\right] \\
&= \mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\, \omega_{\mathbf{k}'} \left(\left[A_{\mathbf{k}}(x), A^\dagger_{\mathbf{k}'}(x')\right] + \left[A_{\mathbf{k}'}(x'), A^\dagger_{\mathbf{k}}(x)\right]\right) \\
&= \mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\, \omega_{\mathbf{k}'} \left(\delta^3(\mathbf{k}-\mathbf{k}') \frac{\mathrm{e}^{-\mathrm{i}k(x-x')}}{2\omega_{\mathbf{k}}} + \delta^3(\mathbf{k}'-\mathbf{k}) \frac{\mathrm{e}^{-\mathrm{i}k'(x'-x)}}{2\omega_{\mathbf{k}'}}\right) \\
&\quad // \text{ expand exp and use } x^0 = x'^0 = t \\
&= \mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\, \omega_{\mathbf{k}'} \left(\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} + \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot(\mathbf{y}-\mathbf{x})}}{2\omega_{\mathbf{k}'}}\right) \delta^3(\mathbf{k}-\mathbf{k}') \\
&= \mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \omega_{\mathbf{k}} \left(\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\omega_{\mathbf{k}}} + \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}}{2\omega_{\mathbf{k}}}\right) \\
&= \mathrm{i} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2} \left(\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} + \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\right) \\
&= \mathrm{i}\frac{1}{2} \left(\delta^3(\mathbf{x}-\mathbf{y}) + \delta^3(\mathbf{y}-\mathbf{x})\right) \\
&= \mathrm{i}\delta^3(\mathbf{y}-\mathbf{x})
\end{aligned}$$

3 The Hamiltonian

We want to evaluate the Hamiltonian

$$H = \int \mathrm{d}^3x\, \mathcal{H}[\phi(x), \pi(x)], \tag{3.1}$$

where the Hamiltonian density is given by,

$$\mathcal{H}[\phi(x), \pi(x)] = \frac{1}{2}\left[\pi^2(x) + (\boldsymbol{\nabla}\phi(x)) \cdot (\boldsymbol{\nabla}\phi(x)) + m^2\phi^2(x)\right]. \tag{3.2}$$

We also have,

$$\begin{aligned}
\phi(x) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(A_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}(x)\right), & (3.3)\\
\pi(x) = \dot{\phi}(x) &= -\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\omega_{\mathbf{k}} \left(A_{\mathbf{k}}(x) - A^\dagger_{\mathbf{k}}(x)\right), & (3.4)\\
\boldsymbol{\nabla}\phi(x) &= \mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\mathbf{k} \left(A_{\mathbf{k}}(x) - A^\dagger_{\mathbf{k}}(x)\right). & (3.5)
\end{aligned}$$

Substituting yields the Hamiltonian density,

$$\begin{aligned}
\mathcal{H}[\phi(x), \pi(x)] &= \frac{1}{2}\left[(\dot{\phi}(x))^2 + (\boldsymbol{\nabla}\phi(x)) \cdot (\boldsymbol{\nabla}\phi(x)) + m^2\phi^2(x)\right]\\
&= \frac{1}{2}\left(-\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\omega_{\mathbf{k}} \left(A_{\mathbf{k}}(x) - A^\dagger_{\mathbf{k}}(x)\right)\right)\left(-\mathrm{i}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\omega_{\mathbf{k}'} \left(A_{\mathbf{k}'}(x) - A^\dagger_{\mathbf{k}'}(x)\right)\right) +\\
&\quad + \frac{1}{2}\left(\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\mathbf{k} \left(A_{\mathbf{k}}(x) - A^\dagger_{\mathbf{k}}(x)\right)\right)\left(\mathrm{i}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\mathbf{k}' \left(A_{\mathbf{k}'}(x) - A^\dagger_{\mathbf{k}'}(x)\right)\right) +\\
&\quad + \frac{1}{2}m^2\left(\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \left(A_{\mathbf{k}}(x) + A^\dagger_{\mathbf{k}}(x)\right)\right)\left(\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}} \left(A_{\mathbf{k}'}(x) + A^\dagger_{\mathbf{k}'}(x)\right)\right).
\end{aligned} \tag{3.6}$$

Take out the integration and suppress the dependency of x,

$$\begin{aligned}
\mathcal{H} = -\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\omega_{\mathbf{k}}\omega_{\mathbf{k}'}\left(A_{\mathbf{k}} - A^\dagger_{\mathbf{k}}\right)\left(A_{\mathbf{k}'} - A^\dagger_{\mathbf{k}'}\right) -\\
-\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\mathbf{k}\cdot\mathbf{k}'\left(A_{\mathbf{k}} - A^\dagger_{\mathbf{k}}\right)\left(A_{\mathbf{k}'} - A^\dagger_{\mathbf{k}'}\right) +\\
+\frac{1}{2}m^2\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\left(A_{\mathbf{k}} + A^\dagger_{\mathbf{k}}\right)\left(A_{\mathbf{k}'} + A^\dagger_{\mathbf{k}'}\right)
\end{aligned} \tag{3.7}$$

We can collect the terms according to different combinations of $A_{\mathbf{k}}(x)$, $A^\dagger_{\mathbf{k}}(x)$,

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\left[-\omega_{\mathbf{k}}\omega_{\mathbf{k}'}\right]\left(A_{\mathbf{k}}A_{\mathbf{k}'} - A_{\mathbf{k}}A^\dagger_{\mathbf{k}'} - A^\dagger_{\mathbf{k}}A_{\mathbf{k}'} + A^\dagger_{\mathbf{k}}A^\dagger_{\mathbf{k}'}\right) +\\
&\quad + \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\left[-\mathbf{k}\cdot\mathbf{k}'\right]\left(A_{\mathbf{k}}A_{\mathbf{k}'} - A_{\mathbf{k}}A^\dagger_{\mathbf{k}'} - A^\dagger_{\mathbf{k}}A_{\mathbf{k}'} + A^\dagger_{\mathbf{k}}A^\dagger_{\mathbf{k}'}\right) +\\
&\quad + \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\, m^2\left(\boxed{A_{\mathbf{k}}A_{\mathbf{k}'}} + A_{\mathbf{k}}A^\dagger_{\mathbf{k}'} + A^\dagger_{\mathbf{k}}A_{\mathbf{k}'} + \boxed{A^\dagger_{\mathbf{k}}A^\dagger_{\mathbf{k}'}}\right), & (3.8)\\
\mathcal{H} &= \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int \frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\left[m^2 - \omega_{\mathbf{k}}\omega_{\mathbf{k}'} - \mathbf{k}\cdot\mathbf{k}'\right]\left(\boxed{A_{\mathbf{k}}A_{\mathbf{k}'} + A^\dagger_{\mathbf{k}}A^\dagger_{\mathbf{k}'}}\right)
\end{aligned}$$

$$+\frac{1}{2}\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\left[m^2+\omega_{\mathbf{k}}\omega_{\mathbf{k}'}+\mathbf{k}\cdot\mathbf{k}'\right]\left(A_{\mathbf{k}}A^\dagger_{\mathbf{k}'}+A^\dagger_{\mathbf{k}}A_{\mathbf{k}'}\right). \tag{3.9}$$

Now, remember that we have $\int \mathrm{d}^3x$ in front of $\mathcal{H}$, where the only x-dependence is in the exponential in $A_{\mathbf{k}}(x)$. Provided that the integrals converge, this integration measure can put inner to $\int \mathrm{d}^3k \int \mathrm{d}^3k'$,

$$H = \boxed{\int \mathrm{d}^3x}\int\frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k'}{(2\pi)^{3/2}}\,[\text{sum of terms with } A_{\mathbf{k}}(x)A_{\mathbf{k}'}(x)\ldots] \tag{3.10}$$

$$= \int \mathrm{d}^3k\int \mathrm{d}^3k'\left[\text{combinations with }\boxed{\frac{1}{(2\pi)^3}\int \mathrm{d}^3x}\,A_{\mathbf{k}}(x)A_{\mathbf{k}'}(x)...\right]. \tag{3.11}$$

We evaluate,

$$\int \mathrm{d}^3x\,A_{\mathbf{k}}(x)A_{\mathbf{k}'}(x) = \int \mathrm{d}^3x\,\frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}}a_{\mathbf{k}}\frac{\mathrm{e}^{-\mathrm{i}k'x}}{\sqrt{2\omega_{\mathbf{k}'}}}a_{\mathbf{k}'} \tag{3.12}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a_{\mathbf{k}}a_{\mathbf{k}'}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}(k+k')x} \tag{3.13}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a_{\mathbf{k}}a_{\mathbf{k}'}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}\mathrm{e}^{\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \tag{3.14}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a_{\mathbf{k}}a_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}\,(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}') \tag{3.15}$$

$$= \frac{(2\pi)^3}{2\omega_{\mathbf{k}}}a_{\mathbf{k}}a_{-\mathbf{k}}\mathrm{e}^{-\mathrm{i}2\omega_{\mathbf{k}}t}\,\delta^3(\mathbf{k}+\mathbf{k}'). \tag{3.16}$$

Also,

$$\int \mathrm{d}^3x\,A^\dagger_{\mathbf{k}}(x)A^\dagger_{\mathbf{k}'}(x) = \int \mathrm{d}^3x\,\frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}}a^\dagger_{\mathbf{k}}\frac{\mathrm{e}^{\mathrm{i}k'x}}{\sqrt{2\omega_{\mathbf{k}'}}}a^\dagger_{\mathbf{k}'} \tag{3.17}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(k+k')x} \tag{3.18}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\int \mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}\mathrm{e}^{-\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \tag{3.19}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}\,(2\pi)^3\delta^3(\mathbf{k}+\mathbf{k}') \tag{3.20}$$

$$= \frac{(2\pi)^3}{2\omega_{\mathbf{k}}}a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}\mathrm{e}^{\mathrm{i}2\omega_{\mathbf{k}}t}\,\delta^3(\mathbf{k}+\mathbf{k}'), \tag{3.21}$$

then,

$$\int \mathrm{d}^3x\,A_{\mathbf{k}}(x)A^\dagger_{\mathbf{k}'}(x) = \int \mathrm{d}^3x\,\frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}}a_{\mathbf{k}}\frac{\mathrm{e}^{\mathrm{i}k'x}}{\sqrt{2\omega_{\mathbf{k}'}}}a^\dagger_{\mathbf{k}'} \tag{3.22}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}(k-k')x} \tag{3.23}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}\mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \tag{3.24}$$

$$= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}\,(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}') \tag{3.25}$$

$$= \frac{(2\pi)^3}{2\omega_{\mathbf{k}}}a_{\mathbf{k}}a^\dagger_{\mathbf{k}}\,\delta^3(\mathbf{k}-\mathbf{k}'), \tag{3.26}$$

and finally,

$$\begin{aligned}
\int \mathrm{d}^3x\, A^\dagger_{\mathbf{k}}(x) A_{\mathbf{k}'}(x) &= \int \mathrm{d}^3x\, \frac{\mathrm{e}^{\mathrm{i}kx}}{\sqrt{2\omega_{\mathbf{k}}}} a^\dagger_{\mathbf{k}} \frac{\mathrm{e}^{-\mathrm{i}k'x}}{\sqrt{2\omega_{\mathbf{k}'}}} a_{\mathbf{k}'} &(3.27)\\
&= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}(k-k')x} &(3.28)\\
&= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t} \mathrm{e}^{-\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} &(3.29)\\
&= \frac{1}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} \mathrm{e}^{\mathrm{i}(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}\, (2\pi)^3 \delta^3(\mathbf{k}-\mathbf{k}') &(3.30)\\
&= \frac{(2\pi)^3}{2\omega_{\mathbf{k}}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\, \delta^3(\mathbf{k}-\mathbf{k}'). &(3.31)
\end{aligned}$$

Now, substituting back into the Hamiltonian we obtain,

$$\begin{aligned}
H = &\frac{1}{2}\int \mathrm{d}^3k \int \mathrm{d}^3k' \left[m^2 - \omega_{\mathbf{k}}\omega_{\mathbf{k}'} - \mathbf{k}\cdot\mathbf{k}'\right] \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \left(A_{\mathbf{k}} A_{\mathbf{k}'} + A^\dagger_{\mathbf{k}} A^\dagger_{\mathbf{k}'}\right) + \\
&+ \frac{1}{2}\int \mathrm{d}^3k \int \mathrm{d}^3k' \left[m^2 + \omega_{\mathbf{k}}\omega_{\mathbf{k}'} + \mathbf{k}\cdot\mathbf{k}'\right] \frac{1}{(2\pi)^3} \int \mathrm{d}^3x \left(A_{\mathbf{k}} A^\dagger_{\mathbf{k}'} + A^\dagger_{\mathbf{k}} A_{\mathbf{k}'}\right), \qquad (3.32)\\
= &\frac{1}{2}\int \mathrm{d}^3k \int \boxed{\mathrm{d}^3k'} \left[m^2 - \omega_{\mathbf{k}}\omega_{\mathbf{k}'} - \mathbf{k}\cdot\mathbf{k}'\right] \times \\
&\times \frac{1}{(2\pi)^3} \left(\frac{(2\pi)^3}{2\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} a_{-\mathbf{k}} \mathrm{e}^{-\mathrm{i}2\omega_{\mathbf{k}}t} + a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} \mathrm{e}^{\mathrm{i}2\omega_{\mathbf{k}}t} \right) \boxed{\delta^3(\mathbf{k}+\mathbf{k}')} \right) + \\
&+ \frac{1}{2}\int \mathrm{d}^3k \int \boxed{\mathrm{d}^3k'} \left[m^2 + \omega_{\mathbf{k}}\omega_{\mathbf{k}'} + \mathbf{k}\cdot\mathbf{k}'\right] \times \\
&\times \frac{1}{(2\pi)^3} \left(\frac{(2\pi)^3}{2\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \right) \boxed{\delta^3(\mathbf{k}-\mathbf{k}')} \right), \qquad (3.33)\\
= &\frac{1}{4}\int \mathrm{d}^3k \left[\cancel{m^2 - \omega^2_{\mathbf{k}} + |\mathbf{k}|^2}\right] \left(\frac{1}{\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} a_{-\mathbf{k}} \mathrm{e}^{-\mathrm{i}2\omega_{\mathbf{k}}t} + a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} \mathrm{e}^{\mathrm{i}2\omega_{\mathbf{k}}t} \right) \right) + \\
&+ \frac{1}{4}\int \mathrm{d}^3k \left[\overbrace{m^2 + \omega^2_{\mathbf{k}} + |\mathbf{k}|^2}^{=2\omega^2_{\mathbf{k}}} \right] \left(\frac{1}{\omega_{\mathbf{k}}} \left(a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \right) \right), \\
= &\frac{1}{2}\int \mathrm{d}^3k\, \omega_{\mathbf{k}} \left(a_{\mathbf{k}} a^\dagger_{\mathbf{k}} + a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \right).
\end{aligned}$$

Finally, since,

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = a_{\mathbf{k}} a^\dagger_{\mathbf{k}'} - a^\dagger_{\mathbf{k}'} a_{\mathbf{k}} = \delta^3(\mathbf{k}-\mathbf{k}'), \qquad (3.34)$$

we have,

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}}] = \delta^3(\mathbf{0}) \qquad (3.35)$$

and consequently the Hamiltonian becomes,

$$H = \int \mathrm{d}^3k\, \omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \boxed{\frac{1}{2}\delta^3(\mathbf{0})} \right). \qquad (3.36)$$

For comparison, if we worked out with the summation over discrete momenta, as in Mandl and Shaw, we would get,

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \boxed{\frac{1}{2}} \right). \qquad \text{MS (3.14)}$$

We see that the last term,

$$\frac{1}{2}\int \mathrm{d}^3k\, \omega_{\mathbf{k}} \delta^3(\mathbf{0}), \quad \text{(or in the discrete version } \frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\text{)}, \tag{3.37}$$

yields an infinite contribution to the expectation value of the vacuum energy $E_0 \to \infty$,

$$E_0 = \langle 0|H|0\rangle = \cancel{\left\langle 0\middle|\int \mathrm{d}^3k\, \omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\middle|0\right\rangle} + \overbrace{\left\langle 0\middle|\frac{1}{2}\int \mathrm{d}^3k\, \omega_{\mathbf{k}} \delta^3(\mathbf{0})\middle|0\right\rangle}^{E_0 \text{ energy of the vacuum}}, \tag{3.38}$$

$$\text{(or in discrete version)} = \cancel{\left\langle 0\middle|\sum_{\mathbf{k}} \omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\middle|0\right\rangle} + \left\langle 0\middle|\frac{1}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}}\middle|0\right\rangle. \tag{3.39}$$

The reason why we get $E_0 \to \infty$ is in ordering ambiguity; in the classical theory we had $aa^* = a^*a$, but here $aa^\dagger = a^\dagger a$. We can avoid this is by constructing the normal ordered Hamiltonian, written $\mathrm{N}(H)$ or $:H:$, defined as,

$$\boxed{\mathrm{N}(H) = :H: = H - \langle 0|H|0\rangle}. \tag{3.40}$$

This ensures that $\langle 0|\,\mathrm{N}(H)\,|0\rangle = 0$. This equivalent to ordering all operators such that creator operators $a^\dagger_{\mathbf{k}}$ are to the left, and annihilation operators $a_{\mathbf{k}}$ are to the right. The normal ordered Hamiltonian is then,

$$\mathrm{N}(H) = \int \mathrm{d}^3k\, \omega_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}. \tag{3.41}$$

Alternativelly, we can start from the normal ordered Lagrangian (assuming the fields to be operators from the beginning)

$$\mathcal{L} = N\left[\frac{1}{2}\left(\partial_\mu \phi(x)\right)\left(\partial^\mu \phi(x)\right) - \frac{1}{2}m^2\phi^2(x)\right]$$

and then arrive at the expansion of the normal ordered Hamiltonian (or other charges) automatically. Whatever approach one uses, at some point the normal ordering must be done during canonical quantization.

N.B. that confusion may arise around the overload of letter N. Namely, $N_{\mathbf{k}}$ is the number operator $N_{\mathbf{k}} := a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ while N() is the normal ordering. The difference is in italic/upright font and in the existence of a subscript for the number operator.

In QFT, we use normal ordering of operators as prescription. This is perfectly valid as only energy differences are observable; hence, any infinite constant of the vacuum is harmless and easily removed by measuring all energies relative to the vacuum state. We will use the normal ordered form of the Hamiltonian extensively when we encounter Wicks theorem. For scattering amplitudes and similar problems, we can use the normal ordered Hamiltonian in the matrix expansion. It does not mean, however, that the infinite offset is not there (e.g., the Casimir effect).

In QFT, there other infinities which can arise: so called divergences. The divergences appear in calculations involving Feynman diagrams with closed loops of virtual particles in them and they affect calculated physical quantities. (The m put in the theory is not the physical (measured) but a *bare* mass.) These are removed by regularization and renormalization.