Tutorial 6, Part 1

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Topics for today

- Gamma matrices, Dirac algebra.

1 Gamma matrices and Dirac algebra

The γ-matrices, also known as the Dirac matrices, are a set of conventional matrices with specific anti-commutation relations that ensure they generate a matrix representation of the Clifford algebra $C\ell_{1,3}(\mathbb{R})$. (Note: It is also possible to define higher-dimensional gamma matrices.)

In Minkowski space, the γ-matrices, $\{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$, are defined by the anti-commutation relations,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \qquad \gamma^{(\mu}\gamma^{\nu)} = \eta^{\mu\nu} \tag{1.1}$$

and the Hermiticity conditions,

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{1.2}$$

As usual, we use the metric to raise/lower (only spacetime, Greek) indices of the γ-matrices, e.g.,

$$\gamma_\mu = \eta_{\mu\nu}\gamma^\nu, \tag{1.3}$$

There is also the fifth-matrix, γ^5, defined by,

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3, \tag{1.4}$$

$$\gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_3\gamma_3. \tag{1.5}$$

N.B. Greek indices will always stand for the values 0-3 and not for 5!

Every γ matrix is a square matrix γ^μ_{ab} with a, b as the spinor indices where $a, b \in \{1, 2, 3, 4\}$ for spin-1/2 fields. Hence, the anti-commutation relations are in fact,

$$\sum_{c=1}^{4} (\gamma^\mu_{ac}\gamma^\nu_{cb} + \gamma^\nu_{ac}\gamma^\mu_{cb}) = 2\eta^{\mu\nu} I_{ab}, \tag{1.6}$$

where I_{ab} is the 4×4 identity matrix.

There are several representations of γ-matrices, the most common takes the form,

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices.

We also define the 4×4 spin matrices,

$$\sigma^{\mu\nu} := \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]. \tag{1.7}$$

The useful notation: **Slash** is defined as

$$\not{a} := a^\mu\gamma_\mu, \qquad \not{a}_{ab} := a_\mu\gamma^\mu_{ab}. \tag{1.8}$$

2 Problems

Problem 1. Show that,

1. $$\sigma^{\mu\nu\dagger} = \gamma^0 \sigma^{\mu\nu} \gamma^0, \tag{2.1}$$

2. $$\gamma^{5\dagger} = \gamma^5 = \gamma_5 = \gamma_5^{-1} \tag{2.2}$$

3. $$\gamma_5 = \frac{\mathrm{i}}{4!} \varepsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = \gamma^5, \tag{2.3}$$

4. $$(\gamma_5)^2 = 1 \tag{2.4}$$

5. $$(\gamma_5 \gamma_\mu)^\dagger = \gamma^0 \gamma_5 \gamma_\mu \gamma^0 \tag{2.5}$$

6. $$\{\gamma_5, \gamma^\mu\} = 0, \tag{2.6}$$

7. $$[\gamma_5, \sigma^{\mu\nu}] = 0, \tag{2.7}$$

8. $$\not{p}^2 = p^2. \tag{2.8}$$

Problem 2. Derive the identities for the following contractions,

1. $$\gamma_\mu \gamma^\mu \tag{2.9}$$

2. $$\gamma_\mu \gamma^\nu \gamma^\mu \tag{2.10}$$

3. $$\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu$$

Problem 3. Derive the following identities for the traces,

1. $$\mathrm{Tr}(\gamma^\mu) \tag{2.11}$$

2. $$\mathrm{Tr}(\gamma_\mu \gamma_\nu) \tag{2.12}$$

3. $$\mathrm{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) \tag{2.13}$$

4.
$$\mathrm{Tr}(\gamma_5)$$

5.
$$\mathrm{Tr}\left(\not{a}_1 \ldots \not{a}_{2n+1}\right)$$

Problem 4. Calculate,

1.
$$\mathrm{Tr}(\not{p}_1 \not{p}_2 ... \not{p}_6) \tag{2.14}$$

2.
$$\gamma_\mu (1 - \gamma_5) \left(\not{p} - m\right) \gamma^\mu \tag{2.15}$$