Tutorial 6 Part 1, Solutions

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1 Problems with solutions

Problem 1. Show that

1.
$$\not{p}^2 = p^2. \tag{1.1}$$

Proposed solution

In this solution, it is used that the four-momenta p_μ and the γ^μ matrices commute. This can be seen by that for each index μ , p_μ is just one component containing a space-time derivative while γ^μ is a 4x4 constant matrix. Moreover, p_μ and p_ν commute with each other (it does not matter in which order two space-time derivatives are perfromed).

$$\not{p}^2 = p_\mu\gamma^\mu p_\nu\gamma^\nu = p_\mu p_\nu\gamma^\mu\gamma^\nu = \frac{1}{2}(p_\mu p_\nu\gamma^\mu\gamma^\nu + p_\nu p_\mu\gamma^\nu\gamma^\mu) =$$
$$= \frac{1}{2}(p_\mu p_\nu\gamma^\mu\gamma^\nu + p_\mu p_\nu\gamma^\nu\gamma^\mu) = \frac{1}{2}p_\mu p_\nu[\gamma^\mu, \gamma^\nu]_+ = \frac{1}{2}p_\mu p_\nu 2\eta^{\mu\nu} = p_\mu p^\mu = p^2$$

Problem 2. Derive the following identities with the contractions,

1.
$$\gamma_\mu\gamma^\mu = 4 \tag{1.2}$$

2.
$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu. \tag{1.3}$$

3.
$$\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4\eta^{\alpha\beta}$$

Proposed solution

1)
$$\gamma_\mu\gamma^\mu = \eta_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}(\eta_{\mu\nu}\gamma^\mu\gamma^\nu + \eta_{\nu\mu}\gamma^\nu\gamma^\mu) = \eta_{\mu\nu}\frac{1}{2}[\gamma^\mu, \gamma^\nu]_+ = \eta_{\mu\nu}\eta^{\mu\nu} = \delta^\mu_\mu = 4.$$

Here it has been used that the metric is symmetric.

2)

$$\begin{aligned}\gamma_\mu\gamma^\nu\gamma^\mu &= \gamma_\mu\left(2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu\right)\\ &= 2\gamma_\mu\eta^{\mu\nu} - \gamma_\mu\gamma^\mu\gamma^\nu\\ &= 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu.\end{aligned}$$

3)

$$\begin{aligned}\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= \gamma_\mu\gamma^\alpha\left(2\eta^{\beta\mu} - \gamma^\mu\gamma^\beta\right)\\ &= 2\eta^{\beta\mu}\gamma_\mu\gamma^\alpha - \boxed{\gamma_\mu\gamma^\alpha\gamma^\mu}\gamma^\beta\\ &= 2\gamma^\beta\gamma^\alpha - (-2\gamma^\alpha)\,\gamma^\beta\\ &= 2\gamma^\beta\gamma^\alpha + 2\gamma^\alpha\gamma^\beta\\ &= 4\eta^{\alpha\beta}.\end{aligned}$$

Problem 3. Derive the following identities with traces,

1.
$$\mathrm{Tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu} \tag{1.4}$$

2.
$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) = 4(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) \tag{1.5}$$

3.
$$\mathrm{Tr}\left(\not{a}_1\ldots\not{a}_{2n+1}\right) = 0$$

Proposed solution

Reminder, we will use the cyclic trace property:

$$\mathrm{Tr}(AB\ldots CD) = \mathrm{Tr}(DAB\ldots C) = \mathrm{Tr}(B\ldots CDA),$$

and also the additivity,

$$\mathrm{Tr}(A+B) = \mathrm{Tr}A + \mathrm{Tr}B.$$

1)

$$\begin{aligned}\mathrm{Tr}(2\eta_{\mu\nu}I) &= \mathrm{Tr}\{\gamma_\mu,\gamma_\nu\}\\ &= \mathrm{Tr}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\\ &= \mathrm{Tr}(\gamma_\mu\gamma_\nu) + \mathrm{Tr}(\gamma_\nu\gamma_\mu)\\ &= \mathrm{Tr}(\gamma_\mu\gamma_\nu) + \mathrm{Tr}(\gamma_\mu\gamma_\nu)\\ &= 2\mathrm{Tr}(\gamma_\mu\gamma_\nu),\\ \eta_{\mu\nu}\mathrm{Tr}(I_4) &= \mathrm{Tr}(\gamma_\mu\gamma_\nu),\\ \mathrm{Tr}(\gamma_\mu\gamma_\nu) &= 4\eta_{\mu\nu}.\end{aligned}$$

2)

$$\begin{aligned}
\mathrm{Tr}\left(\boxed{\gamma_\mu\gamma_\nu}\gamma_\rho\gamma_\sigma\right) &= \mathrm{Tr}\left(\left(2\eta_{\mu\nu}-\gamma_\nu\gamma_\mu\right)\gamma_\rho\gamma_\sigma\right)\\
&= \mathrm{Tr}\left(2\eta_{\mu\nu}\gamma_\rho\gamma_\sigma-\gamma_\nu\gamma_\mu\gamma_\rho\gamma_\sigma\right)\\
&= 2\eta_{\mu\nu}\mathrm{Tr}\left(\gamma_\rho\gamma_\sigma\right)-\mathrm{Tr}\left(\gamma_\nu\boxed{\gamma_\mu\gamma_\rho}\gamma_\sigma\right)\\
&= 2\eta_{\mu\nu}\mathrm{Tr}\left(\gamma_\rho\gamma_\sigma\right)-\mathrm{Tr}\left(\gamma_\nu\left(2\eta_{\mu\rho}-\gamma_\rho\gamma_\mu\right)\gamma_\sigma\right)\\
&= 2\eta_{\mu\nu}\mathrm{Tr}\left(\gamma_\rho\gamma_\sigma\right)-2\eta_{\mu\rho}\mathrm{Tr}\left(\gamma_\nu\gamma_\sigma\right)+\mathrm{Tr}\left(\gamma_\nu\gamma_\rho\boxed{\gamma_\mu\gamma_\sigma}\right)\\
&= 2\eta_{\mu\nu}\mathrm{Tr}\left(\gamma_\rho\gamma_\sigma\right)-2\eta_{\mu\rho}\mathrm{Tr}\left(\gamma_\nu\gamma_\sigma\right)+2\eta_{\mu\sigma}\mathrm{Tr}\left(\gamma_\nu\gamma_\rho\right)-\\
&\qquad -\mathrm{Tr}\left(_{\uparrow}\gamma_\nu\gamma_\rho\gamma_\sigma\underset{\leftarrow}{\gamma_\mu}\right).\\
\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma) &= 4(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}).
\end{aligned}$$

3) From $\gamma_5^2=1$ and $\{\gamma_5,\gamma^\mu\}=0$,

$$\begin{aligned}
\mathrm{Tr}\left(\not{a}_1\ldots\not{a}_{2n+1}\right) &= \mathrm{Tr}\left(\gamma_5\gamma_5\not{a}_1\not{a}_2\ldots\not{a}_{2n+1}\right)\\
&= -\mathrm{Tr}\left(\gamma_5\not{a}_1\gamma_5\not{a}_2\ldots\not{a}_{2n+1}\right)\\
&= \mathrm{Tr}\left(\gamma_5\not{a}_1\not{a}_2\gamma_5\ldots\not{a}_{2n+1}\right)\\
&= -\mathrm{Tr}\left(\gamma_5\not{a}_1\not{a}_2\ldots\not{a}_{2n+1}\gamma_5\right)\\
&= -\mathrm{Tr}\left(\not{a}_1\not{a}_2\ldots\not{a}_{2n+1}\right).
\end{aligned}$$

Problem 4. Calculate,

1.
$$\mathrm{Tr}(\not{p}_1\not{p}_2\cdots\not{p}_6) \tag{1.6}$$

2.
$$\gamma_\mu(1-\gamma_5)\left(\not{p}-m\right)\gamma^\mu \tag{1.7}$$

Proposed solution

1) After a lengthy calculation,

$$\begin{aligned}
&\mathrm{Tr}(\not{a}_1\not{a}_2\cdots\not{a}_6) =\\
&= 4\Big\{(a_1\cdot a_2)\left[(a_3\cdot a_4)(a_5\cdot a_6)-(a_3\cdot a_5)(a_4\cdot a_6)+(a_3\cdot a_6)(a_4\cdot a_5)\right]-\\
&\quad -(a_1\cdot a_3)\left[(a_2\cdot a_4)(a_5\cdot a_6)-(a_2\cdot a_5)(a_4\cdot a_6)+(a_2\cdot a_6)(a_4\cdot a_5)\right]+\\
&\quad +(a_1\cdot a_4)\left[(a_2\cdot a_3)(a_5\cdot a_6)-(a_2\cdot a_5)(a_3\cdot a_6)+(a_2\cdot a_6)(a_3\cdot a_5)\right]-\\
&\quad -(a_1\cdot a_5)\left[(a_2\cdot a_3)(a_4\cdot a_6)-(a_2\cdot a_4)(a_3\cdot a_6)+(a_2\cdot a_6)(a_3\cdot a_4)\right]+\\
&\quad +(a_1\cdot a_6)\left[(a_2\cdot a_3)(a_4\cdot a_5)-(a_2\cdot a_4)(a_3\cdot a_5)+(a_2\cdot a_5)(a_3\cdot a_4)\right]\Big\}
\end{aligned}$$

2)

$$
\begin{aligned}
\gamma_\mu(1-\gamma_5)\left(\not{p}-m\right)\gamma^\mu &= \gamma_\mu(1-\gamma_5)\left(p^\sigma\gamma_\sigma - m\right)\gamma^\mu \\
&= p^\sigma\gamma_\mu(1-\gamma_5)\gamma_\sigma\gamma^\mu - m\gamma_\mu(1-\gamma_5)\gamma^\mu \\
&= p^\sigma(\gamma_\mu\gamma_\sigma\gamma^\mu - \boxed{\gamma_\mu\gamma_5}\gamma_\sigma\gamma^\mu) - m(\gamma_\mu\gamma^\mu - \gamma_\mu\boxed{\gamma_5\gamma^\mu}) \\
&= p^\sigma\boxed{(\gamma_\mu\gamma_\sigma\gamma^\mu} + \gamma_5\boxed{\gamma_\mu\gamma_\sigma\gamma^\mu}) - m(\boxed{\gamma_\mu\gamma^\mu} + \boxed{\gamma_\mu\gamma^\mu}\gamma_5) \\
&= p^\sigma(-2\gamma_\sigma - 2\gamma_5\gamma_\sigma) - m(4+4\gamma_5) \\
&= -2(1+\gamma_5)\not{p} - 4(1+\gamma_5)mI \\
&= -(1+\gamma_5)(2\not{p}+4m).
\end{aligned}
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