# Solutions for Tutorial 7, HT2016 on the Dirac field

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1. We have the Dirac equation

$$[i\hbar\gamma^\mu\partial_\mu - mc]\psi(x) = 0\,. \tag{1}$$

The left hand side of the Dirac equation under transformation becomes

$$[i\gamma^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu - m]S\psi(x) = \tag{2}$$

$$= SS^{-1}[i\gamma^\mu(\Lambda^{-1})^\nu_\mu\partial_\nu - m]S\psi(x)\,. \tag{3}$$

Since $S$ is constant, it can be moved through the partial derivative. The matrix $S$ acts on spinors while $(\Lambda^{-1})^\nu_\mu$ acts on spacetime, so these two matrices commute and we can rewrite the expression as:

$$LHS = S[iS^{-1}\gamma^\mu S(\Lambda^{-1})^\nu_\mu\partial_\nu - m]\psi(x)\,. \tag{4}$$

We now rearragne the expression

$$\gamma^\nu = \Lambda^\nu_\mu S\gamma^\mu S^{-1} \tag{5}$$

by multiplying it by $S^{-1}$ from the left and by $S$ from the right, giving:

$$S^{-1}\gamma^\nu S = \Lambda^\nu_\mu\gamma^\mu \tag{6}$$

Then

$$\begin{aligned} LHS = S[i\Lambda^\mu_\sigma\gamma^\sigma(\Lambda^{-1})^\nu_\mu\partial_\nu - m]\psi(x) = \\ S[i\delta^\nu_\sigma\gamma^\sigma\partial_\nu - m]\psi(x) = \\ S[i\gamma^\nu\partial_\nu - m]\psi(x) = 0 \rightarrow \\ (i\gamma^\nu\partial_\nu - m)\psi(x) = 0 \end{aligned} \tag{7}$$

We have shown that the Dirac equation is Lorentz invariant.

2. For $\psi^\dagger(x)\psi(x)$:

$$\psi^\dagger(x)\psi(x) \rightarrow \psi^\dagger(x)S^\dagger S\psi(x)\,. \tag{8}$$

This quantity is not Lorentz invariant, since $S^\dagger \neq S^{-1}$ ($S$ is not unitary). We now evaluate the transformation $\bar{\psi}(x)\psi(x)$:

$$\bar{\psi}(x)\psi(x) \to \psi^\dagger(x)S^\dagger\gamma^0 S\psi(x)\,. \tag{9}$$

This can be done by rearranging the identity

$$S^{-1} = \gamma^0 S^\dagger \gamma^0 \tag{10}$$

by multpilying it from the right and left by $\gamma^0$:

$$\gamma^0 S^{-1}\gamma^0 = S^\dagger. \tag{11}$$

Then Eq. (9) can be written as:

$$\begin{aligned}\bar{\psi}(x)\psi(x) \to \psi^\dagger\gamma^0 S^{-1}\gamma^0\gamma^0 S\psi(x) = \\ = \psi^\dagger\gamma^0 S^{-1} S\psi(x) = \bar{\psi}(x)\psi(x)\,.\end{aligned} \tag{12}$$

So $\bar{\psi}(x)\psi(x)$ is Lorentz invariant provided the anti-commutation relations of the $\gamma^\mu$ matrices and given Eq. (10).

3. Choose the following representation for the $\gamma^\mu$ matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{14}$$

Here $\sigma_k$ are the Pauli spin matrices. In matrix form and using natural units, the Dirac equation in this representation is:

$$\begin{pmatrix} i\frac{\partial}{\partial x^0} - m & i\frac{\partial}{\partial x^j}\sigma_j \\ -i\frac{\partial}{\partial x^j}\sigma_j & -i\frac{\partial}{\partial x^0} - m \end{pmatrix}\psi(x) = 0\,. \tag{15}$$

Assume the plane wave solution $\psi(x) = u_r(\mathbf{p})\mathrm{e}^{-ipx}$. At this stage, in order to have a general solution, we allow both positive and negative energies ($p_0$) in the exponent. Below this notation is used: $\sigma_j p_j = -\sigma \cdot \mathbf{p}$, with the - sign originating from $\eta^{ij} = -1$, employed to raise the index of $p_j$. Equation (15) then becomes:

$$\begin{pmatrix} E - m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E - m \end{pmatrix} u_r(\mathbf{p})\mathrm{e}^{-ipx} = 0\,. \tag{16}$$

The sign on $E$ has contributions from $i^2$ and from the minus sign in the exponent. For the space-derivatives, there is a + sign in the exponent, hence the relative sign change for the off-diagonal terms. A non-trivial solution only exists if the determinant of the matrix is zero:

$$-(E-m)(E+m)+(\sigma\cdot\mathbf{p})^2=0\,. \tag{17}$$

The following identity is used to evaluate this expression:

$$(\sigma\cdot\mathbf{p})^2=|\mathbf{p}|^2. \tag{18}$$

It can be derived with the by now familar trick:

$$\begin{aligned}(\sigma\cdot\mathbf{p})^2=\sigma_i p_i\sigma_j p_j=\frac{1}{2}(\sigma_i p_i\sigma_j p_j+\sigma_j p_j\sigma_i p_i)=\\ =\frac{1}{2}p_i p_j[\sigma_i,\sigma_j]_+=\frac{1}{2}p_i p_j 2\delta_{ij}=p_i p_i=|\mathbf{p}|^2.\end{aligned} \tag{19}$$

So, from setting the determinant to zero, we get

$$-E^2+m^2+|\mathbf{p}|^2=0\,, \tag{20}$$

or

$$E=\pm\sqrt{m^2+|\mathbf{p}|^2}=\pm E_{\mathbf{p}}\,. \tag{21}$$

First solve for the positive energy $E=+E_{\mathbf{p}}$. Write $u_r(\mathbf{p})$ on the form:

$$u_r(\mathbf{p})=\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \tag{22}$$

where $\phi$ and $\chi$ have two components each. Choosing the second row in Eq. (16) gives:

$$(\sigma\cdot\mathbf{p})\phi-(E_{\mathbf{p}}+m)\chi=0\,, \tag{23}$$

which gives

$$\chi=\frac{\sigma\cdot\mathbf{p}}{E_{\mathbf{p}}+m}\phi\,. \tag{24}$$

Then the solution is of the form:

$$u_r(\mathbf{p})=N\begin{pmatrix}\phi\\ \frac{\sigma\cdot\mathbf{p}}{E_{\mathbf{p}}+m}\phi\end{pmatrix}, \tag{25}$$

where $N$ is the normalization. Two orhogonal solutions that satisty this condition can be constructed (one for $r=1$ and one for $r=2$).

Now we look at the negative energy solutions $v_r(\mathbf{p})=u_r(-E_{\mathbf{p}},-\mathbf{p})$ that appear in the combination $\psi=v_r(\mathbf{p})\mathrm{e}^{ipx}$. For this, the solution

$E = -E_{\mathbf{p}}$ and the replacement $\mathbf{p} \to -\mathbf{p}$ in Eq. (16) are used. This time choosing the first row of the modified Eq. (16) then gives:

$$-(E_{\mathbf{p}} + m)\phi + \sigma \cdot \mathbf{p}\chi = 0 \to \phi = \frac{\sigma \cdot \mathbf{p}}{E_{\mathbf{p}} + m}\chi\,. \tag{26}$$

So,

$$v_r(\mathbf{p}) = N \begin{pmatrix} \frac{\sigma \cdot \mathbf{p}}{E_{\mathbf{p}}+m}\chi \\ \chi \end{pmatrix} \tag{27}$$

Again, we can choose orthogonal (and orthonormal) solutions for $v_r(\mathbf{p})$.

In the non-relativistic limit $\mathbf{p} \to 0$, $u_r(\mathbf{p})$ can be described by the top two components ($\phi$) while $v_r(\mathbf{p})$ can be described by the bottom two components ($\chi$). In the ultrarelativistic limit, however, $m \sim 0$, $E_{\mathbf{p}} \sim |\mathbf{p}|$ and the contributions from $\phi$ and $\chi$ is equally large to both $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$.

The normalization $N$ can be found by normalizing $\bar{u}_r(\mathbf{p})u_s(\mathbf{p}) = \delta_{rs}$, $\bar{v}_r(\mathbf{p})v_s(\mathbf{p}) = -\delta_{rs}$, choosing $\phi_r^\dagger \phi_s = \chi_r^\dagger \chi_s = \delta_{rs}$ and using the explicit form of $\gamma^0$ to write $\bar{u}_r(\mathbf{p})$ as $u_r^\dagger(\mathbf{p})\gamma^0$ and $\bar{v}_r(\mathbf{p})$ as $v_r^\dagger(\mathbf{p})\gamma_0$. The result is $N = \sqrt{\frac{E_{\mathbf{p}}+m}{2m}}$.

Using the explicit forms of $\bar{u}_r(\mathbf{p})$, $u_r(\mathbf{p})$ and $\gamma^0$, as written above, one arrives at

$$1 = \bar{u}_r(\mathbf{p})u_r(\mathbf{p}) = N^2(\phi^\dagger\phi - \left(\frac{\sigma \cdot \mathbf{p}}{E_{\mathbf{p}} + m}\right)^2 \phi^\dagger\phi) \tag{28}$$

Using the relation shown in Eq. (37),

$$N^2 = \frac{1}{1 - \frac{(E_{\mathbf{p}}+m)(E_{\mathbf{p}}-m)}{(E_{\mathbf{p}}+m)^2}} = \frac{E_{\mathbf{p}} + m}{E_{\mathbf{p}} + m - E_{\mathbf{p}} + m} = \frac{E_{\mathbf{p}} + m}{2m} \tag{29}$$

4. We have

$$\Lambda_\pm(\mathrm{p}) = \frac{\pm\not{p} + m}{2m} \tag{30}$$

Compute $\Lambda_\pm(\mathrm{p})^2$:

$$\Lambda_+(\mathrm{p})^2 = \frac{(\not{p} + m)^2}{(2m)^2} = \frac{\not{p}^2 + 2\not{p}m + m^2}{4m^2}\,. \tag{31}$$

Use $\not{p}^2 = p^2 = m^2$:

$$\Lambda_+(\mathrm{p})^2 = \frac{2m^2 + 2\not{p}m}{4m^2} = \frac{2m(\not{p} + m)}{4m^2} = \frac{\not{p} + m}{2m} = \Lambda_+(\mathrm{p})\,. \tag{32}$$

Similarly,

$$\Lambda_-(\text{p})^2 = \frac{(-\not{p}+m)^2}{(2m)^2} = \frac{(m^2 - 2\not{p}m + m)^2}{4m^2} = \frac{-\not{p}+m}{2m} = \Lambda_-(\text{p})\,. \quad (33)$$

Now compute $\Lambda_+(\text{p})\Lambda_-(\text{p})$:

$$\Lambda_+(\text{p})\Lambda_-(\text{p}) = \frac{\not{p}+m}{2m}\cdot\frac{-\not{p}+m}{2m} = \frac{m^2-\not{p}^2}{4m^2} = 0\,. \quad (34)$$

We have shown that $\Lambda_\pm(\text{p})^2 = \Lambda_\pm(\text{p})$ and that $\Lambda_+(\text{p})\Lambda_-(\text{p}) = 0$, as it should be for projection operators. These projection operators are used to separate the positive energy and negative energy solutions from linear combinations of the four solutions $u_r(\mathbf{p})$, $v_r(\mathbf{p})$.

5. We are to show that $\sum u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{\not{p}+m}{2m}$ and $\sum v_r(\mathbf{p})\bar{v}_r(\mathbf{p}) = \frac{\not{p}-m}{2m}$. We are to use the explicit form for $u_r(\mathbf{p})$ found in excercise 4 (before choosing the coordinate system), the same convention for the $\gamma^\mu$ matrices and $(\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m})^\dagger = \frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m}$. The **outer product** of the spinors is evaluated:

$$\sum_{r=1}^{2} u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \sum_{r=1}^{2} N^2 \begin{pmatrix} \phi_r \\ \frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m}\phi_r \end{pmatrix} \begin{pmatrix} \phi_r^\dagger & \phi_r^\dagger\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \frac{E_\mathbf{p}+m}{2m}\sum_{r=1}^{2} \begin{pmatrix} \phi_r\phi_r^\dagger & -\phi_r\phi_r^\dagger\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m} \\ \frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m}\phi_r\phi_r^\dagger & -\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m}\phi_r\phi_r^\dagger\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m} \end{pmatrix} \quad (35)$$

Using completeness relations $\phi_1\phi_1^\dagger + \phi_2\phi_2^\dagger = I_2$ (outer product) gives

$$\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{E_\mathbf{p}+m}{2m}\begin{pmatrix} 1 & -\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m} \\ \frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m} & -\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m}\frac{\sigma\cdot\mathbf{p}}{E_\mathbf{p}+m} \end{pmatrix} =$$

$$= \frac{1}{2m}\begin{pmatrix} E_\mathbf{p}+m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -(E_\mathbf{p}-m) \end{pmatrix} = \frac{1}{2m}\begin{pmatrix} E_\mathbf{p}+m & -\sigma\cdot\mathbf{p} \\ \sigma\cdot\mathbf{p} & -E_\mathbf{p}+m \end{pmatrix}. \quad (36)$$

The following has been used to simplify the expression for the element on row 2, column 2:

$$\sigma\cdot\mathbf{p}\cdot\sigma\cdot\mathbf{p} = |\mathbf{p}|^2 = E_\mathbf{p}^2 - m^2 = (E_\mathbf{p}+m)(E_\mathbf{p}-m)\,. \quad (37)$$

Using the representation of $\gamma^\mu$ as defined above,

$$\sum_r u_r(\mathbf{p})\bar{u}_r(\mathbf{p}) = \frac{\not{p}+m}{2m}\,. \quad (38)$$

In a similar way, it can be found that

$$\sum_r v_r(\mathbf{p})\bar{v}_r(\mathbf{p}) = \frac{\not{p} - m}{2m} \,. \tag{39}$$

These properties are frequently used in computing cross sections.

**Note**: $\bar{u}_r(\mathbf{p})u_r(\mathbf{p})$ (row x column vector) is shorthand notation for the inner product $\bar{u}_r(\mathbf{p})_\alpha u_r(\mathbf{p})^\alpha$, which gives a number. $u_r(\mathbf{p})\bar{u}_r(\mathbf{p})$ (column x row vector) is shorthand notation for the outer product $u_r(\mathbf{p})_\alpha \bar{u}_r(\mathbf{p})_\beta$, which gives a matrix. Using this formalism, the RHS also gets indices $\left(\frac{\not{p}-m}{2m}\right)_{\alpha\beta}$. The indices $\alpha$ and $\beta$ give the components of the matrix.

If you write out the indices $\alpha, \beta$, the order is not important, that is $\bar{u}_r(\mathbf{p})_\alpha u_r(\mathbf{p})_\beta = u_r(\mathbf{p})_\beta \bar{u}_r(\mathbf{p})_\alpha$. Contraction over the same index denotes the scalar product: $\bar{u}_r(\mathbf{p})_\alpha u_r(\mathbf{p})^\alpha = u_r(\mathbf{p})^\alpha \bar{u}_r(\mathbf{p})_\alpha =$ just one component. If you drop these indices, however (as above), the order is important.

6. To show that the Dirac equation satisfies the Klein-Gordon equation, multiply the Dirac equation from the left by the complex conjugate of the quantity within the parenthesis:

$$(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \,. \tag{40}$$

Evaluating the LHS and equating it to the RHS:

$$\begin{aligned} LHS = (\gamma^\nu \gamma^\mu \partial_\mu \partial_\nu + m^2)\psi(x) = (\partial_\mu \partial_\nu \frac{1}{2}(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu) \\ (\eta^{\mu\nu}\partial_\mu \partial_\nu + m^2)\psi(x) = (\partial^2 + m^2)\psi(x) = 0 \end{aligned} \tag{41}$$

It has been used that $\partial\mu$, $\partial\nu$ commute and the dummy indices have been renamed one of the terms, together with anticommutation properties for $\gamma$ matrices. Thus, we get back the Klein-Gordon equation.

7. *Hint rather than problem*

The property ${u_r}^\dagger(\mathbf{p})v_s(-\mathbf{p}) = 0$ can be useful when evaluation expressions containing the Dirac field. If you get something similar, see if you can manipulate the expression to bring it to the desired form.