Tutorial 9

Topics

- The Schrödinger, Heisenberg and Interaction Pictures
- Perturbation Expansion of the S-Matrix
- Wick's Theorem

1 The Pictures: Schrödinger, Heisenberg and Interaction

References: Section 1.5 on p. 20 in Mandl and Shaw, Section 2.2 on p. 80 in Sakuari, Section 5.5 on p. 336 in Sakurai

The Schrödinger, Heisenberg and Interaction pictures are three different ways of describing the time development of a system. Quantities in these three pictures are often distinguished by the labels 'S', 'H' and 'I' (N.b. the roman upright letters).

The Heisenberg picture (HP) and the Schrödinger picture (SP) differ only by a basis change with respect to time-dependency, which is the difference between active and passive transformations. Time dependence is ascribed to quantum states in the Schrödinger picture and to operators in the Heisenberg picture. In the interaction picture (IP), a complicated Hamiltonian H has a natural decomposition into a simple "free" Hamiltonian H_0 and a perturbation in form of the potential H_{int}. The following table summarizes how states and observales change with respect to different pictures:

	Schrödinger picture	**Heisenberg picture**	**Interaction picture**
State ket	Evolution determined by H	No change	Evolution determined by H_{int}
Observable	No change	Evolution determined by H	Evolution determined by H_0

Until now we have treated the free-fields (i.e. non-interacting fields) using the Heisenberg picture, in which state vectors are constant in time and the operators carry the full time dependence. Here we shall employ the interaction picture which leads to two essential simplifications:

Firstly, in the IP, the operators satisfy the Heisenberg-like equations of motion, but involving the free Hamiltonian H_0 only, not the complete Hamiltonian H. Secondly, *if the interaction Lagrangian density $\mathcal{L}_{\text{int}}$ does not involve derivatives, the fields canonically conjugate to the interacting fields and to the free fields are identical.* Since the IP and the HP are related by a unitary transformation, it follows that in the IP, the interacting fields satisfy the same commutation relations as the free fields.

In the interaction picture in general, the system is described by a time-dependent state vector $|\Phi(t)\rangle^{\mathrm{I}}$, which satisfies the equation of motion

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle^{\mathrm{I}} = H^{\mathrm{I}}_{int}(t)\,|\Phi(t)\rangle^{\mathrm{I}}\,, \qquad \text{MS (6.12)}$$

where the interaction Hamiltonian in the IP reads

$$H^{\mathrm{I}}_{\mathrm{int}}(t) = \mathrm{e}^{\mathrm{i}H_0(t-t_0)}H^{\mathrm{S}}_{\mathrm{int}}\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}$$

Herein $H^{\mathrm{S}}_{\mathrm{I}}$ and $H^{\mathrm{I}}_0 = H^{\mathrm{S}}_0$ are the interaction and free-field Hamiltonians in the Schrödinger picture, respectively.

Let $\mathcal{O}$ be a field operator (e.g. ϕ, ψ, A^μ...) and $|\Phi(t)\rangle^{\mathrm{I}}$ be a state (the configuration of all the fields is inside $|\Phi(t)\rangle^{\mathrm{I}}$.) In the interaction picture we have:

$$\begin{gathered}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{O}^{\mathrm{I}}(t) = [\mathcal{O}^{\mathrm{I}}(t), H^{\mathrm{I}}_0], \quad \text{equivalent to the free-field equations,}\\
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle^{\mathrm{I}} = H^{\mathrm{I}}_{\mathrm{int}}(t)\,|\Phi(t)\rangle^{\mathrm{I}}\,, \quad \text{governs evolution of states,}\\
|\Phi(t_0)\rangle^{\mathrm{S}} = |\Phi(t_0)\rangle^{\mathrm{H}} = |\Phi(t_0)\rangle^{\mathrm{I}}\,.
\end{gathered}$$

- At $t = t_0 \to -\infty$, we have the initial state $|i\rangle = |\Phi(-\infty)\rangle^{\mathrm{I}}$.
- At $t \to \infty$, we have all the possible final states as $|\Phi(\infty)\rangle^{\mathrm{I}}$. Note that this is not the same as the particular final state $|f\rangle$ that we are interested in ($|f\rangle$ is only the part of $|\Phi(\infty)\rangle^{\mathrm{I}}$; the latter can be very complicated at high energy collisions, even starting with very simple $|i\rangle = |\Phi(-\infty)\rangle^{\mathrm{I}}$).

The solution, in general, can be written as:

$$|\Phi(\infty)\rangle^{\mathrm{I}} = S\,|\Phi(-\infty)\rangle^{\mathrm{I}}\,, \quad |\Phi(\infty)\rangle^{\mathrm{I}} = S\,|i\rangle\,,$$

where S is obtained by solving

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle^{\mathrm{I}} = H^{\mathrm{I}}_{\mathrm{int}}(t)\,|\Phi(t)\rangle^{\mathrm{I}}\,.$$

Consider a scattering process from some initial particles $|i\rangle$, we get some final particles as $|f\rangle$. The projection of the final state $|f\rangle$ in the all possible final states is given by

$$\langle f|\Phi(t_0)\rangle\,.$$

This is *the contribution of* $|f\rangle$ *in* $|\Phi(\infty)\rangle^{\mathrm{I}}$. Thus, the probability that $|\Phi(\infty)\rangle^{\mathrm{I}}$ contains the final state $|f\rangle$ is given by,

$$|\langle f|\Phi(t_0)\rangle|^2 = |\langle f|S|i\rangle|^2\,.$$

The matrix S is called the scattering matrix, and has the matrix elements

$$S_{fi} = \langle f|S|i\rangle \,.$$

We want to compute the elements of the scattering matrix (S-matrix). If we started from $\langle i|i\rangle = 1$, we want also ${}^{\mathrm{I}}\langle\Phi(\infty)|\Phi(\infty)\rangle^{\mathrm{I}} = 1$, meaning that $\left\langle f\middle|S^\dagger S\middle|i\right\rangle = 1$, i.e. the S-matrix must be unitary matrix, $S^\dagger S = \hat{1}$.[1]

Now, consider the Lagrangian density of QED which reads [with the **normal ordering** assumed!

$$\mathcal{L}^{\mathrm{QED}} = \mathrm{i}\bar{\psi}\left(\gamma^\mu D_\mu\psi - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu = \partial_\mu + \mathrm{i}qA_\mu.$$

The Lagrangian density can be written [see also MS Eqs. (4.66)-(4.68), (11.8b)]

$$\mathcal{L} = \underbrace{\underbrace{\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi}_{\mathcal{L}_0^{\mathrm{D}}} \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\mathcal{L}_0^{\mathrm{M}}}}_{\mathcal{L}_0 = \mathcal{L}_0^{\mathrm{D}} + \mathcal{L}_0^{\mathrm{M}}} \underbrace{-q\bar{\psi}\gamma^\mu A_\mu\psi}_{\mathcal{L}_{\mathrm{int}} = -s^\mu A_\mu} , \qquad \text{MS (6.9)}$$

that is, it can be split into the free-field piece $\mathcal{L}_0 = \mathcal{L}_0^{\mathrm{D}} + \mathcal{L}_0^{\mathrm{M}}$ and the interaction piece $\mathcal{L}_{\mathrm{int}}$

$$\begin{aligned} \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{int}}, \qquad \mathcal{L}_0 &= \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \text{MS (6.8)} \\ \mathcal{L}_{\mathrm{int}} &= -q\bar{\psi}\gamma^\mu A_\mu\psi = \boxed{e\bar{\psi}\not{A}\psi}, \\ s^\mu &= q\bar{\psi}\gamma^\mu\psi = -e\bar{\psi}\gamma^\mu\psi. \end{aligned}$$

N.b. $q = -e$ for electrons! After going to Hamiltonian formalism, we have the canonical momenta as before we had for $\mathcal{L}_0$, since

$$\frac{\partial\mathcal{L}_{\mathrm{I}}}{\partial\dot{\psi}} = 0, \quad \text{and} \quad \frac{\partial\mathcal{L}_{\mathrm{I}}}{\partial\dot{A}_\mu} = 0,$$

Consequently the Hamiltonian reads

$$\begin{aligned} H &= \int \mathrm{d}^3x \left(\pi^a\dot{\phi}_a - \mathcal{L}\right) = H_0 + H_{\mathrm{int}}\,, \qquad \text{MS (6.11)} \\ H_{\mathrm{int}} &= \int \mathrm{d}^3x\,\mathcal{H}_{\mathrm{int}}, \quad \text{where} \quad \mathcal{H}_{\mathrm{int}} = -\mathcal{L}_{\mathrm{int}}\,. \end{aligned}$$

or explicitly, with the normal ordering

$$\mathcal{H}_{\mathrm{int}}^{\mathrm{I}} = q\,\mathrm{N}\left[\bar{\psi}\gamma^\mu A_\mu\psi\right]^{\mathrm{I}} = \boxed{-e\,\mathrm{N}\left[\bar{\psi}\not{A}\psi\right]^{\mathrm{I}}}, \qquad \text{MS (6.24)}$$

[1] Sometimes it is not obvious that S-matrix is unitary. We can use unitarity to deduce some properties of the evolution equations.

2 Perturbation Expansion of the S-matrix

(See also 4.2 Perturbation Expansion of Correlation Functions in Peskin and Schroeder)

$$S = \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_{-\infty}^{t_n=\infty} \mathrm{d}t_{n-1} \int_{-\infty}^{t_{n-1}} \mathrm{d}t_{n-2} \ldots \int_{-\infty}^{t_2} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_0 \left[H_{\mathrm{int}}^{\mathrm{I}}(t_{n-1}) \ldots H_{\mathrm{int}}^{\mathrm{I}}(t_1) H_{\mathrm{int}}^{\mathrm{I}}(t_0) \right]$$

In the above expression for the S-matrix, the structure of the integral ensures $t_n \geq t_{n-1} \geq \cdots \geq t_1 \geq t_0$, that is, the product $H_{\mathrm{int}}^{\mathrm{I}}(t_{n-1}) \ldots H_{\mathrm{int}}^{\mathrm{I}}(t_1) H_{\mathrm{int}}^{\mathrm{I}}(t_0)$ is automatically time ordered.

To simplify integration, we can change the upper integration bound to be all ∞ by **imposing time ordering of the integrand**. In such case, we are covering $n!$-times the same volume $\mathrm{d}^n x$; this we compensate by introducing the factor $1/n!$

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_{n-1} \int_{-\infty}^{\infty} \mathrm{d}t_{n-2} \ldots \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_0 \mathrm{T} \left\{ H_{\mathrm{int}}^{\mathrm{I}}(t_{n-1}) \ldots H_{\mathrm{int}}^{\mathrm{I}}(t_1) H_{\mathrm{int}}^{\mathrm{I}}(t_0) \right\}$$

The above integrals are from lecture notes. The equivalent integrals from M&S are,

$$S = \sum_{n=0}^{\infty} (-\mathrm{i})^n \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{n-1}} \mathrm{d}t_n \left[H_{\mathrm{int}}^{\mathrm{I}}(t_1) H_{\mathrm{int}}^{\mathrm{I}}(t_2) \ldots H_{\mathrm{int}}^{\mathrm{I}}(t_n) \right] \qquad \text{MS (6.22a)}$$

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_n \mathrm{T} \left\{ H_{\mathrm{int}}^{\mathrm{I}}(t_1) H_{\mathrm{int}}^{\mathrm{I}}(t_2) \ldots H_{\mathrm{int}}^{\mathrm{I}}(t_n) \right\} \qquad \text{MS (6.22b)}$$

Now, to obtain the explicitly covariant S-matrix expansion, we rewrite the integral in terms of the interaction Hamiltonian density,

$$S = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \ldots \int \mathrm{d}^4 x_n \mathrm{T} \left\{ \mathcal{H}_{\mathrm{I}}(x_1) \mathcal{H}_{\mathrm{I}}(x_2) \ldots \mathcal{H}_{\mathrm{I}}(x_n) \right\}, \qquad \text{MS (6.23)}$$

where the S-matrix in the last line is given as the sum of the various perturbation orders of $S^{(n)}$, which, by taking $\mathcal{H}_{\mathrm{int}}^{\mathrm{I}}(x) = -\mathcal{L}_{\mathrm{int}}^{\mathrm{I}}(x)$ can be rewritten more nicely in terms of Lagrangian densities (Eq. (7.1) in M&S),

$$S = \sum_{n=0}^{\infty} S^{(n)}, \quad S^{(n)} = \frac{\mathrm{i}^n}{n!} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \ldots \int \mathrm{d}^4 x_n \mathrm{T} \left\{ \mathcal{L}_{\mathrm{int}}^{\mathrm{I}}(x_1) \mathcal{L}_{\mathrm{int}}^{\mathrm{I}}(x_2) \ldots \mathcal{L}_{\mathrm{int}}^{\mathrm{I}}(x_n) \right\}$$

where, (note ϕ^{I} are the solutions to the free-theory),

$$\mathcal{L}_{\mathrm{int}}^{\mathrm{I}}(x) := \mathcal{L}_{\mathrm{int}}[\phi^{\mathrm{I}}](x).$$

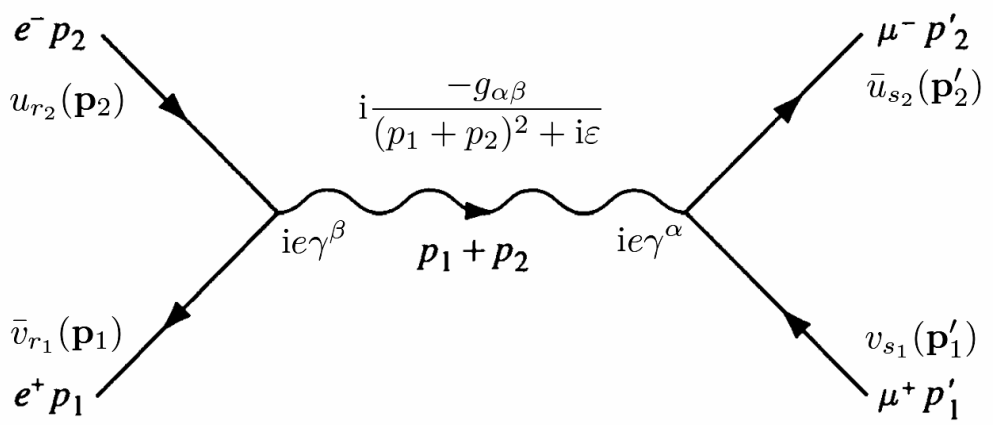
For example, for QED,

$$\begin{aligned}\mathcal{H}^{\mathrm{I}}_{\mathrm{int}}(x) &= -e\,\mathrm{N}\left[\bar{\psi}(x)\not{A}(x)\psi(x)\right]^{\mathrm{I}} \qquad \text{MS (7.2)}\\ &= -e\,\mathrm{N}\left[\left(\bar{\psi}^{+}+\bar{\psi}^{-}\right)_x\left(\not{A}^{+}+\not{A}^{-}\right)_x\left(\psi^{+}+\psi^{-}\right)_x\right]^{\mathrm{I}}\end{aligned}$$

The last interaction gives rise to eight basic processes, e.g. the term $\mathrm{N}(\bar{\psi}^{+}A^{-}\psi^{+})_x$ corresponds to the annihilation of an electron-positron pair with the creation of a photon at x. The individual positive/negative frequency operators are summarized in the following table.

Table 1: Operators figuring in $\mathcal{H}^{\mathrm{I}}_{\mathrm{int}}(x) = -e\mathrm{N}\left[\bar{\psi}(x)\not{A}(x)\psi(x)\right]^{\mathrm{I}}$

$A(x)$	$A^{+}(x)$	photon	absorption γ	$\varepsilon^{\mu}_{r}(\mathbf{k})$	$a_r(\mathbf{k})$	e^{-ikx}
	$A^{-}(x)$		creation γ	$\varepsilon^{\mu}_{r}(\mathbf{k})$	$a^{\dagger}_r(\mathbf{k})$	e^{ikx}
$\psi(x)$	$\psi^{+}(x)$	electron	absorption e^{-}	$u_{r\alpha}(\mathbf{p})$	$c_r(\mathbf{p})$	$\mathrm{e}^{-ipx/\hbar}$
	$\psi^{-}(x)$	positron	creation e^{+}	$v_{r\alpha}(\mathbf{p})$	$d^{\dagger}_r(\mathbf{p})$	$\mathrm{e}^{ipx/\hbar}$
$\bar{\psi}(x)$	$\bar{\psi}^{+}(x)$	positron	absorption e^{+}	$\bar{v}_{r\alpha}(\mathbf{p})$	$d_r(\mathbf{p})$	$\mathrm{e}^{-ipx/\hbar}$
	$\bar{\psi}^{-}(x)$	electron	creation e^{-}	$\bar{u}_{r\alpha}(\mathbf{p})$	$c^{\dagger}_r(\mathbf{p})$	$\mathrm{e}^{ipx/\hbar}$



We can use Wick's theorem for writing the time-orderd terms in an S-matrix expansion as a sum of normal products.

Note: The n-th order term n the S-matrix expansion contains a factor $1/n!$ and n integration variables $x_1, ..., x_n$. These are only summation variables and can be attached to the n vertices of a given Feynman graph in n ways. We can omit the factor $1/n!$ if we consider only topologically different Feynman diagrams, i.e. diagrams which differ only in the labelling of vertices are considered the same.

3 Wick's Theorem

Wick's theorem is essential in evaluating scattering amplitudes and S-matrix expansions. The theorem is simple to state.

The basic ingredients are: (1) $\mathrm{T}\{\dots\}$, (2) $\mathrm{N}(\dots)$ and (3) $\underbrace{AB}$.

1. The first is the time ordering operator $\mathrm{T}\{\dots\}$

$$\mathrm{T}\{A(x)B(y)\} = \theta(x_0 - y_0)A(x)B(y) + \theta(y_0 - x_0)B(y)A(x) \ .$$

If A and B are fermions, there is a minus sign instead of a plus sign in between.

2. $\mathrm{N}(AB)$ is the normal ordering, such that

$$\mathrm{N}(AB) = A^-B^+ + A^-B^- + A^+B^+ + B^-A^+ \, .$$

Here we expanded the field in positive and negative frequency parts, as $A = A^+ + A^-$ and $B = B^+ + B^-$, where $A^+|0\rangle = 0$.
For two operators, the following relation holds:

$$AB = \mathrm{N}(AB) + \left[A^+, B^-\right]$$

3. The last object $\overbrace{AB}$ is a contraction between the field A and B, defined as,

$$\overbrace{AB} := \langle 0|\mathrm{T}\{AB\}|0\rangle \ .$$

This object is i times the propagator, which is $\Delta_{\mathrm{F}}(x-y)$ for Klein-Gordon bosons, $S_{\mathrm{F}}(x-y)$ for Dirac fermions and $D_{\mathrm{F}}(x-y)$ for vector bosons[2].

Wick's theorem states that for any two operators,

$$\mathrm{T}\{AB\} = \mathrm{N}(AB) + \overbrace{AB} \, . \tag{3.1}$$

We can verify this by taking the vacuum expectation value on both sides of (3.1). We see that $\langle 0|\overbrace{AB}|0\rangle = \overbrace{AB}$, and that $\langle 0|\mathrm{N}(AB)|0\rangle = 0$ verifies (3.1). For three operators, the expansion is,

$$T\{ABC\} = \mathrm{N}(ABC) + \mathrm{N}(\overbrace{AB}C) + \mathrm{N}(\overbrace{ABC}) + \mathrm{N}(A\overbrace{BC}) \, . \tag{3.2}$$

We see that it is the normal ordering and all possible ways of doing contractions between

[2] In perturbation theory, usually we want to calculate the *two-point correlation function*, or *two-point Green's function*, $\langle\Omega|T\{A(x)B(y)\}|\Omega\rangle$ where $|\Omega\rangle$ is the ground state of the interacting theory, which is different from $|0\rangle$. The correlation function can be interpreted physically as the amplitude for propagation of a particle or excitation between y and x. In the free theory, it is simply the Feynman propagator $\langle 0|T\{A(x)B(y)\}|0\rangle_{\text{Free}}$.

the different terms. For 4 operators we similarly have,

$$\begin{aligned} T\{ABCD\} &= \mathrm{N}(ABCD) + \mathrm{N}(\overbracket{A}BCD) + \mathrm{N}(\overbracket{AB}CD) + \mathrm{N}(\overbracket{ABC}D) + \mathrm{N}(\overbracket{ABCD}) \\ &\quad + \mathrm{N}(A\overbracket{B}CD) + \mathrm{N}(A\overbracket{BC}D) + \mathrm{N}(A\overbracket{BCD}) \\ &\quad + \mathrm{N}(\overbracket{A}B\underbracket{C}D) + \mathrm{N}(\overbracket{A}\underbracket{BC}D) + \mathrm{N}(\overbracket{A}\underbracket{BCD}). \end{aligned} \tag{3.3}$$

As we can see it is not only single contractions that are taken. All possible ways of making all possible number of contractions should be considered. Now the contractions are just complex numbers so we can move them out of the normal ordering. However doing to we need to keep track of how many other operators are passed to create the contraction. For bosons this is unimportant, but for fermions we need to add a minus sign for every passed operators. Thus, if $ABCD$ are all bosons, we can write (3.3) as,

$$\begin{aligned} \mathrm{T}\{ABCD\} &= \mathrm{N}(ABCD) + \mathrm{N}(CD)\,\overbracket{AB} + \mathrm{N}(BD)\,\overbracket{AC} + \mathrm{N}(BC)\,\overbracket{AD} \\ &\quad + \mathrm{N}(AD)\,\overbracket{BC} + \mathrm{N}(AC)\,\overbracket{BD} + \mathrm{N}(AB)\,\overbracket{CD} \\ &\quad + \overbracket{AB}\underbracket{CD} + \overbracket{AC}\underbracket{BD} + \overbracket{AD}\underbracket{BC} \end{aligned} \tag{3.4}$$

whereas for $ABCD$ all fermions we would get,

$$\begin{aligned} \mathrm{T}\{ABCD\} &= \mathrm{N}(ABCD) + \mathrm{N}(CD)\,\overbracket{AB} - \mathrm{N}(BD)\,\overbracket{AC} + \mathrm{N}(BC)\,\overbracket{AD} \\ &\quad + \mathrm{N}(AD)\,\overbracket{BC} - \mathrm{N}(AC)\,\overbracket{BD} + \mathrm{N}(AB)\,\overbracket{CD} \\ &\quad + \overbracket{AB}\underbracket{CD} - \overbracket{AC}\underbracket{BD} + \overbracket{AD}\underbracket{BC}. \end{aligned} \tag{3.5}$$

Finally, note that *contracted things are not measurable*.

3.1 Filtering Contractions

Of course, not all kins of operators have valid contractions. In fact only if A contains operators that to not commute with operators in B, will the contraction exist.

For instance for a *boson* ϕ and a *fermion* ψ we have

$$\overbracket{\phi\psi} = 0\,.$$

The same goes for *bosons of different types* ϕ, φ,

$$\underbracket{\phi\varphi} = 0\,.$$

Also for fermions we have,

$$\overbracket{\psi\psi} = \underbracket{\bar{\psi}\bar{\psi}} = 0\,.$$

Contractions that survive are,

$$\begin{aligned}
\overbracket{\phi\phi} &= \mathrm{i}\Delta_{\mathrm{F}}, && \textbf{neutral scalar bosons},\\
\overbracket{\phi^\star\phi} = \overbracket{\phi\phi^\star} &= \mathrm{i}\Delta_{\mathrm{F}}, && \textbf{charged scalar bosons},\\
\overbracket{\psi\bar{\psi}} = -\overbracket{\bar{\psi}\psi} &= \mathrm{i}S_{\mathrm{F}}, && \textbf{fermions},\\
\overbracket{AA} &= \mathrm{i}D_{\mathrm{F}}, && \textbf{vector bosons}.
\end{aligned}$$

With this in mind, what are the Wick expansions of,

$$\mathrm{T}\left\{\phi_x\bar{\psi}_y\right\}, \quad \mathrm{T}\left\{\phi_x\bar{\psi}_y\psi_z\right\}, \quad \mathrm{T}\left\{\phi_x\phi_y\psi_z\right\}, \quad \mathrm{T}\left\{\phi_{x_1}\phi_{x_2}\phi_{x_3}\right\}, \quad \mathrm{T}\left\{\varphi_{x_1}\varphi_{x_2}\phi_{y_1}\phi_{y_2}\right\} \ ?$$

Here $O_x \equiv O\left(x\right)$, and also ϕ and φ are neutral bosons. For instance, the last expression expands

$$\begin{aligned}
\mathrm{T}\left\{\varphi_{x_1}\varphi_{x_2}\phi_{y_1}\phi_{y_2}\right\} &= \mathrm{N}(\varphi_{x_1}\varphi_{x_2}\phi_{y_1}\phi_{y_2}) + \mathrm{N}(\phi_{y_1}\phi_{y_2})\overbracket{\varphi_{x_1}\varphi_{x_2}}\\
&\quad +\mathrm{N}(\varphi_{x_1}\varphi_{x_2})\overbracket{\phi_{y_1}\phi_{y_2}} + \overbracket{\varphi_{x_1}\varphi_{x_2}}\underbracket{\phi_{y_1}\phi_{y_2}}
\end{aligned}$$

so only 4 out of 9 terms survived.

3.2 Time Ordering of Normal Ordered Products

Now, since the time ordered produces we encounter will come from an expansion of the operator

$$S = \mathrm{T}\left\{e^{\mathrm{i}\int d^4x\mathcal{H}_I(x)}\right\},$$

where $\mathcal{H}_I\left(x\right)$ is some normal ordered interaction term. In QED $\mathcal{H} = \mathrm{i}eN\left(\bar{\psi}\not{A}\psi\right)$, and in ϕ^4 theory it would be $\mathcal{H} = \frac{\lambda}{4!}\mathrm{N}\left(\phi^4\right)$. This will give time ordered products of the type,

$$\mathrm{T}\left\{\mathcal{H}(x)\cdot\mathcal{H}(y)\cdot\cdots\cdot\mathcal{H}(z)\right\}.$$

In these cases, there is an extension to Wicks theorem that states that **only contractions between different groups of normal ordered operators need to be considered**.

As an example we have,

$$\begin{aligned}
\mathrm{T}\left\{\mathrm{N}\left(AB\right)\cdot\mathrm{N}\left(CD\right)\right\} &= \mathrm{N}\left(ABCD\right) + \mathrm{N}(\overbracket{AB\cdot CD}) + \mathrm{N}(\overbracket{AB\cdot CD})\\
&\quad +\mathrm{N}(A\overbracket{B\cdot C}D) + \mathrm{N}(A\overbracket{B\cdot CD})\\
&\quad +\mathrm{N}(\overbracket{A\underbracket{B\cdot C}D}),
\end{aligned}$$

3.3 The Second Order Terms $S^{(2)}$ in QED

We now turn to the more physical situation of

$$\mathrm{T}\left\{ \mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x) \cdot \mathrm{N}(\bar{\psi}_y \not{A}_y \psi_y) \right\}.$$

This is the second order expansion of the QED interaction,

$$\begin{aligned} S^{(2)} &= \frac{\mathrm{i}^2}{2!} \int \mathrm{d}^4x \mathrm{d}^4y \, \mathrm{T}\left\{ e\mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x) \cdot e\mathrm{N}(\bar{\psi}_y \not{A}_y \psi_y) \right\} \\ &= -\frac{e^2}{2} \int \mathrm{d}^4x \mathrm{d}^4y \, \mathrm{T}\left\{ \mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x) \, \mathrm{N}(\bar{\psi}_y \not{A}_y \psi_y) \right\} \end{aligned}$$

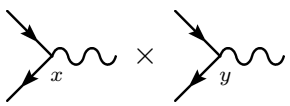
$$S^{(2)} = \sum_{i=\mathrm{A}}^{\mathrm{F}} S_i^{(2)} \qquad \text{MS (7.4)}$$

Lets find all terms contained in this sum. With the corresponding S-matrix element boxed, we have,

$$\mathrm{T}\left\{\mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x) \cdot \mathrm{N}(\bar{\psi}_y \not{A}_y \psi_y)\right\} = \qquad \text{MS (7.5)}$$

$$\begin{aligned}
& \boxed{S_{\mathrm{A}}^{(2)}} && \mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y \psi_y) \\
\text{1-contractions:} \quad & \boxed{S_{\mathrm{B}}^{(2)}} && + \mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y} \not{A}_y \psi_y) \\
& \boxed{S_{\mathrm{B}}^{(2)}} && + \mathrm{N}(\bar{\psi}_x \not{A}_x \overbrace{\psi_x \bar{\psi}_y} \not{A}_y \psi_y) \\
& \boxed{S_{\mathrm{C}}^{(2)}} && + \mathrm{N}(\bar{\psi}_x \overbrace{\not{A}_x \psi_x \bar{\psi}_y} \not{A}_y \psi_y) \\
\text{2-contractions:} \quad & \boxed{S_{\mathrm{D}}^{(2)}} && + \mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y} \not{A}_y \psi_y) \\
& \boxed{S_{\mathrm{D}}^{(2)}} && + \mathrm{N}(\bar{\psi}_x \not{A}_x \overbrace{\psi_x \bar{\psi}_y} \not{A}_y \psi_y) \\
& \boxed{S_{\mathrm{E}}^{(2)}} && + \mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y} \not{A}_y \psi_y) \\
\text{3 -contractions:} \quad & \boxed{S_{\mathrm{F}}^{(2)}} && + \mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \overbrace{\psi_x \bar{\psi}_y} \not{A}_y \psi_y}).
\end{aligned}$$

In terms of Feynman diagrams: These should be interpreted as:

$\mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y \psi_y) \quad \sim$  $\times$

This represents the creation and/or annihilation of two fermions and a photon, at two disconnected and unrelated places in the universe. At this level, the directions of the different legs are not determined. This is since all three of $\bar{\psi}$, ψ and $\not{A}$ all contain both positive an negative frequency parts: This mean that particles can both be created an destroyed at the point x. Thus one of these diagrams contains a total of 8 processes:

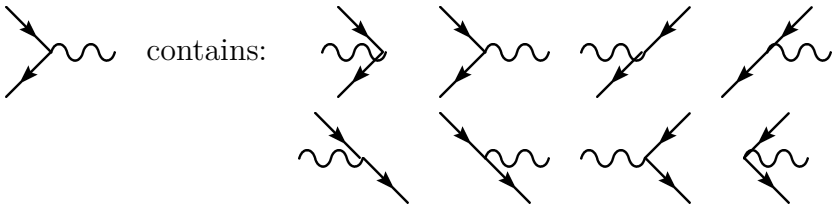
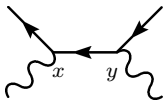


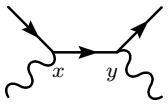
Figure 1: Processes of $\bar{\psi}_x \not{A}_x \psi_x$ in the configuration space at the position x

3.3.1 1 -contractions $S_{\mathrm{B}}^{(2)}$, $S_{\mathrm{C}}^{(2)}$

$$\mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y}\psi_y) \quad \sim$$

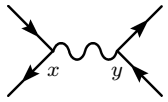


$$\mathrm{N}(\bar{\psi}_x \not{A}_x \overbrace{\psi_x \bar{\psi}_y} \not{A}_y \psi_y) \quad \sim$$



These two diagrams are the same only that the direction of charge is running in different directions. These can represent electron-photon scattering, but also electron-positron annihilation into two photons is determined by this diagram.

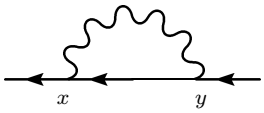
$$\mathrm{N}(\bar{\psi}_x \not{A}_x \overbrace{\psi_x \bar{\psi}_y \not{A}_y} \psi_y) \quad \sim$$



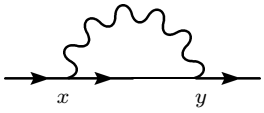
This diagram is mainly electron-electron scattering and electron-positron scattering. The diagram written is electron-positron scattering by fusing to an intermediate virtual photon.

3.3.2 2 -contractions $S_{\mathrm{D}}^{(2)}$, $S_{\mathrm{E}}^{(2)}$

$$\mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y \psi_y}) \quad \sim$$

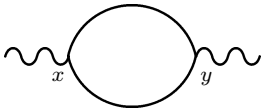


$$\mathrm{N}(\bar{\psi}_x \not{A}_x \overbrace{\psi_x \bar{\psi}_y} \not{A}_y \psi_y) \quad \sim$$



These diagrams are fermions propagating in opposite directions, else the same. We cannot observe (distinguish) these two from ⟵} and ⟶}, respectively. However, we can measure their effect as they contribute to the mass of the electron (the process is called self-energy corrections). Loops give infinities and this infinities are added to the *bare* (unphysical) mass of the electron (= the electron mass you put into the theory is called bare mass and it's unphysical). The calculated mass is 'real' measured mass.

$$\mathrm{N}(\overbrace{\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y} \not{A}_y \psi_y) \quad \sim$$

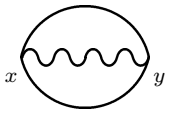


This is a photon propagator. The photon self-energy does not produce the mass because we have the gauge invariant theory.

All of these terms represent higher order contributions to the bare propagators $\longrightarrow$ and $\sim\!\!\sim$. It is the business of renormalization[3] to account for these propagator in a correct manner. In this course, we shall mostly do things without loops.

3.3.3 3 -contractions $S_{\mathrm{F}}^{(2)}$

$$\mathrm{N}(\bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y \psi_y) \quad \sim$$



This is a vacuum bubble. It has no external legs, so it does not contribute to any scattering diagrams (except those where nothing happens at all).

The vacuum bubbles are important in other parts of the theory.

4 A note on Feynman Diagrams and the Summary Factor

The following examples are stolen from Peskin and Schroeder, Section 4.3.

Now, consider the ϕ^4 theory with the interaction term

$$\mathcal{L}_{\mathrm{I}} = -\frac{\lambda}{4!}\phi^4$$

having the Hamiltonian

$$H = H_0 + H_{\mathrm{I}} = H_{\mathrm{KG}} + \int \mathrm{d}^3x \, \frac{\lambda}{4!}\phi^4(x).$$

As we have seen, Wick's theorem allows us to turn any expression of the form

$$\langle 0|T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\}|0\rangle$$

into a sum of products of Feynman propagators.

$$\begin{aligned} T\{\phi(x_1)\phi(x_2)\cdots\phi(x_n)\} \; &= \\ &= \; N\{\phi(x_1)\phi(x_2)\cdots\phi(x_n) \; + \text{ all possible contractions}\,\} \end{aligned}$$

In particular (denoting $\phi_i \equiv \phi(x_i)$)

[3] *Regularization*: we render infinities to become finite. *Renormalization*: we remove infinities (reassign values to masses, charges). *Running couplings and masses*: they are studied in the subject "renormalization group equations" (Wilson RG equations, got Nobel prize for this work). QFT and statistical mechanics are unified by Wilson RG (explains the real physics behind QFT).

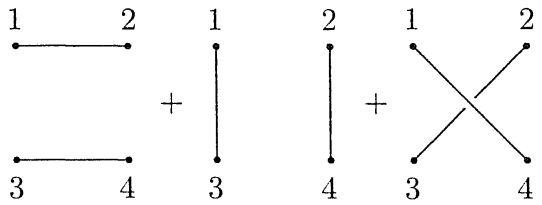
$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \phi_1\overbrace{\phi_2\phi_3}\phi_4 + \phi_1\overbrace{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4} + \overbrace{\phi_1\phi_2\overbrace{\phi_3\phi_4}}\}.$$

$$N\{\overbrace{\phi_1\phi_2}\phi_3\phi_4\} \qquad \text{means} \qquad D_F(x_1 - x_3)\cdot N\{\phi_2\phi_4\}.$$

$$\begin{aligned}\langle 0|\, T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle = {} & D_F(x_1-x_2)D_F(x_3-x_4) \\ & + D_F(x_1-x_3)D_F(x_2-x_4) \\ & + D_F(x_1-x_4)D_F(x_2-x_3).\end{aligned}$$

Now, consider our case of four fields all at different spacetime points. Let us represent each of the points x_1 through x_4 by a node, and each factor $D_F(x-y)$ by a line joining x to y. Then the last equation can be represented as the sum of three diagrams (the configuration-space Feynman diagrams)

$\langle 0|\, T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle =$



Things get more interesting when the expression contains more than one field at the same spacetime point.

$$\langle 0|\, T\left\{\phi(x)\phi(y)\,(-\mathrm{i})\int \mathrm{d}t \int \mathrm{d}^3z\,\frac{\lambda}{4!}\phi^4\right\}|0\rangle =$$
$$= \langle 0|\, T\left\{\phi(x)\phi(y)\,(-\mathrm{i}\frac{\lambda}{4!})\int \mathrm{d}t \int \mathrm{d}^3z\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle$$

Now, apply Wick's theorem. We get one term for every way of contracting the six ϕ operators with each other in pairs. There are 15 ways to do this, but (fortunately) only two of them are really different. If we contract $\phi(x)$ with $\phi(y)$, then there are three ways to contract the four $\phi(z)$'s with each other, and all three give identical expressions. The other possibility is to contract $\phi(x)$ with one of the $\phi(z)$ (four choices), $\phi(y)$ with one of the others (three choices), and the remaining two $\phi(z)$'s with each other (one choice). There are twelve ways to do this, and all give identical expressions. Thus we have:

$$\begin{aligned}\langle 0|\, T\left\{\phi(x)\phi(y)\,(-\mathrm{i})\int \mathrm{d}t \int \mathrm{d}^3z\frac{\lambda}{4!}\phi^4\right\}|0\rangle \; = \\ = \; 3\times(-\mathrm{i}\frac{\lambda}{4!})D_{\mathrm{F}}(x-y)\int \mathrm{d}^4z\, D_{\mathrm{F}}(z-z)\, D_{\mathrm{F}}(z-z) + \\ + 12\times(-\mathrm{i}\frac{\lambda}{4!})\int \mathrm{d}^4z\, D_{\mathrm{F}}(x-z)\, D_{\mathrm{F}}(y-z)\, D_{\mathrm{F}}(z-z)\end{aligned}$$

The Summary Factor. In practice one always draws the diagram first, using it as a mnemonic device for writing down the analytic expression. But then the question arises. What is the overall constant? We could, of course, work it out as above: We could associate a factor $(-\mathrm{i}\lambda/n!)\int \mathrm{d}z^4$ with each vertex, put in the $1/n!$ from the Taylor series, and then do the combinatorics by writing out the product of fields and counting. But the $1/n!$ from the Taylor series will almost always cancel the $n!$ from interchanging the vertices, so we can just forget about both of these factors. Furthermore, the generic vertex has four lines coming in from four different places, so the various placements of these contractions into $\phi\phi\phi\phi$ generates a factor of 4!, which cancels the denominator in $(-\mathrm{i}\lambda/4!)$. It is therefore conventional to associate the expression $(-\mathrm{i}\lambda)\int \mathrm{d}z^4$ with each vertex. (This was the reason for the factor of 4! in the ϕ^4 coupling, i.e., $\lambda\phi^4/4!$)