Tutorial 11

FK8027 - Quantum Field Theory

Monday 28$^{\text{th}}$ January, 2019

Topics for today

- The relativistic definition of flux
- The cross section
- The spin-sums lemma
- e^-e^+ production in electromagnetic field

1 The relativistic definition of flux

In M&S, the differential cross-section is defined in eq. (8.8) to be

$$\mathrm{d}\sigma := \frac{w}{\phi}\Pi_f \frac{V\mathrm{d}^3 p_f}{(2\pi)^3}, \tag{1}$$

where we assume to know the rate of events $w = \dfrac{\mathrm{d}|S_{fi}|^2}{\mathrm{d}t}$, and the flux of the particles in the initial state ϕ. Equivalently, we can define the cross-section as [LLH75, p. 34]

$$\mathrm{d}N = \sigma\phi\mathrm{d}V\mathrm{d}t, \tag{2}$$

where $\mathrm{d}N$ is the number of events in $\mathrm{d}V\mathrm{d}t$. In non-relativistic mechanics, the flux is equal to

$$\phi = n_1 n_2 v_{\mathrm{rel}}, \tag{3}$$

where n_1 and n_2 are the number densities of the particles and v_{rel} is the modulus of the relative velocity between them, i.e.,

$$v_{\mathrm{rel}} = |\vec{v}_1 - \vec{v}_2|. \tag{4}$$

We need to extend the definition of the flux to the relativistic case. This is not trivial as it can seem, because $\mathrm{d}w$ must be invariant under Lorentz transformations, and both n_i and the relativistic v_{rel} are not. However,, if the target particles are at rest, then we do not have to compose any velocity and $v_{\mathrm{rel}} = |\vec{v}_1|$. In this case, then, the classical formula can be directly extended,

$$\mathrm{d}N = \sigma n_1 n_2 v_{\mathrm{rel}}\mathrm{d}V\mathrm{d}t, \tag{5}$$

with the *relativistic* $v_{\rm rel} = |\vec{v}_1|$. We need an expression which reduces to (5) when the target is at rest, and is Lorentz invariant. Suppose that this expression can be written as,

$$\mathrm{d}N = (An_1n_2)\,\mathrm{d}V\mathrm{d}t. \tag{6}$$

We know that $\mathrm{d}V\mathrm{d}t = \mathrm{d}^4x$ is invariant under Lorentz transformations, and $\mathrm{d}N$, being a pure number, is invariant as well. This implies that An_1n_2 *has to be invariant.* Let's see how n_i transforms under a Lorentz transformation.

The number of particles in a given volume is a Lorentz invariant, so we have

$$n\mathrm{d}V = n'\mathrm{d}V', \tag{7}$$

where the primes indicate quantities in the new coordinates. We know, from the length contraction, that the volume transforms as

$$\mathrm{d}V' = \frac{\mathrm{d}V}{\gamma}, \tag{8}$$

where we introduced the relativistic γ factor. Therefore,

$$n\mathrm{d}V = n'\mathrm{d}V' = n'\frac{\mathrm{d}V}{\gamma} \Longrightarrow n' = \gamma n. \tag{9}$$

Hence, the quantity An_1n_2 can be written as

$$An_1n_2 = A\gamma_1 n_1^0\gamma_2 n_2^0 = A\frac{E_1}{m_1}\frac{E_2}{m_2}n_1^0n_2^0, \tag{10}$$

where n_1^0, n_2^0 are the number densities in the rest frames of the targets and the projectiles, respectively. Now n_1^0, n_2^0, m_1, m_2 are invariant under Lorentz transformations, se we want to impose that AE_1E_2 is invariant as well. In addition, if AE_1E_2 is an invariant, such is $AE_1E_2/\left(p_1^\mu p_{2\mu}\right)$, since $p_1^\mu p_{2\mu}$ is itself an invariant. Let's call it I,

$$I := \frac{AE_1E_2}{p_1^\mu p_{2\mu}} = \frac{AE_1E_2}{E_1E_2 - \vec{p}_1\cdot\vec{p}_2}. \tag{11}$$

Let's go to the rest frame of particle 2, i.e., $\vec{p}_2 = 0$,

$$I = \frac{AE_1m_2}{E_1m_2} = A. \tag{12}$$

We know from (5) that in this frame we should get $\sigma v_{\rm rel}$, therefore we have determined the invariant I,

$$I = \sigma v_{\rm rel}, \tag{13}$$

which does not change under Lorentz transformations. It follows that

$$A = \frac{p_1^\mu p_{2\mu}}{E_1 E_2} I = \frac{p_1^\mu p_{2\mu}}{E_1 E_2} \sigma v_{\text{rel}}, \tag{14}$$

and

$$\mathrm{d}N = \sigma v_{\text{rel}} \frac{p_1^\mu p_{2\mu}}{E_1 E_2} n_1 n_2 \mathrm{d}V \mathrm{d}t. \tag{15}$$

At this point we need to determine the relativistic v_{rel}. The following formulas hold in special relativity [LLH75, p. 35], [Can17], [Tsa10, Sec. 6.4],

$$\begin{aligned} v_{\text{rel}} &= \left[\left(p_1^\mu p_{2\mu}\right)^2 - m_1^2 m_2^2 \right]^{1/2} \frac{1}{p_1^\mu p_{2\mu}} \\ &= \left[(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \wedge \vec{v}_2)^2 \right]^{1/2} \frac{1}{1 - \vec{v}_1 \cdot \vec{v}_2}, \end{aligned} \tag{16a}$$

$$p_1^\mu p_{2\mu} = E_1 E_2 \left(1 - \vec{v}_1 \cdot \vec{v}_2\right), \tag{16b}$$

which imply

$$E_1 E_2 v_{\text{rel}} = \left[\left(p_1^\mu p_{2\mu}\right)^2 - m_1^2 m_2^2 \right]^{1/2} \frac{1}{1 - \vec{v}_1 \cdot \vec{v}_2}, \tag{17a}$$

$$E_1 E_2 \left[(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \wedge \vec{v}_2)^2 \right]^{1/2} = \left[\left(p_1^\mu p_{2\mu}\right)^2 - m_1^2 m_2^2 \right]^{1/2}. \tag{17b}$$

This formula show that eq. (8.9) in M&S in not precise. They are calling v_{rel} the quantity $\left[(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \wedge \vec{v}_2)^2 \right]^{1/2}$, which is not the relativistic relative velocity. We call this quantity v_{sep}, the "separation" between the relativistic velocities (it reduces to the relative velocity in the non-relativistic case),

$$v_{\text{sep}} := \left(1 - \vec{v}_1 \cdot \vec{v}_2\right) v_{\text{rel}} = \left[(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \wedge \vec{v}_2)^2 \right]^{1/2}. \tag{18}$$

Using these formulas, one obtains,

$$\begin{aligned} \frac{\mathrm{d}N}{n_1 n_2} &= \sigma v_{\text{rel}} \frac{p_1^\mu p_{2\mu}}{E_1 E_2} \mathrm{d}V \mathrm{d}t \\ &= \sigma \left[(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \wedge \vec{v}_2)^2 \right]^{1/2} \frac{1}{1 - \vec{v}_1 \cdot \vec{v}_2} \frac{E_1 E_2 \left(1 - \vec{v}_1 \cdot \vec{v}_2\right)}{E_1 E_2} \mathrm{d}V \mathrm{d}t \\ &= \sigma \left[(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \wedge \vec{v}_2)^2 \right]^{1/2} \mathrm{d}V \mathrm{d}t = \sigma v_{\text{sep}} \mathrm{d}V \mathrm{d}t. \end{aligned} \tag{19}$$

We now can finally define the relativistic flux as follows

$$\mathrm{d}N = \sigma v_{\text{rel}} \frac{p_1^\mu p_{2\mu}}{E_1 E_2} n_1 n_2 \mathrm{d}V \mathrm{d}t = \sigma v_{\text{rel}} \frac{E_1 E_2 \left(1 - \vec{v}_1 \cdot \vec{v}_2\right)}{E_1 E_2} n_1 n_2 \mathrm{d}V \mathrm{d}t$$

$$= \sigma \left[n_1 n_2 \left(1 - \vec{v}_1 \cdot \vec{v}_2\right) v_{\text{rel}} \right] \mathrm{d}V \mathrm{d}t, \tag{20}$$

$$\phi \coloneqq n_1 n_2 \left(1 - \vec{v}_1 \cdot \vec{v}_2\right) v_{\text{rel}} = n_1 n_2 \, v_{\text{sep}} \xrightarrow{\vec{v}_{1,2} \to 0} n_1 n_2 v_{\text{rel}}, \tag{21}$$

which is what we are looking for, compatibly with (5).

You can easily accept that the quantity v_{rel} in M&S cannot be a relativistic relative velocity if you look at eq. (8.10a), which can become grater than 1 very easily (see also [Wei05, pp. 137–139]). Hence, eq. (8.9) in M&S is correct if we replace the subscript "rel" with "sep". To quote [Tsa10, p. 169], "in Special Relativity nothing is obvious and everything has to be calculated explicitly".[1]

At this point, we can define a four-vector $\mathcal{J}_i$ for the particle i as [Can17]

$$\mathcal{J}_i \coloneqq \left(n_i, n_i \vec{v}\right), \tag{22}$$

such that the flux is given by,

$$\phi = \left(\mathcal{J}_1 \cdot \mathcal{J}_2\right) v_{\text{rel}} = n_1 n_2 \left(1 - \vec{v}_1 \cdot \vec{v}_2\right) v_{\text{rel}}. \tag{23}$$

As the last comment, we note that for collinear particles $\vec{v}_1 \wedge \vec{v}_2 = 0$, hence the relative velocity is just

$$v_{\text{rel}} = \frac{\left|\vec{v}_1 - \vec{v}_2\right|}{1 - \vec{v}_1 \cdot \vec{v}_2} = \frac{v_{\text{sep}}}{1 - \vec{v}_1 \cdot \vec{v}_2}. \tag{24}$$

M&S are considering collinear particles, indeed v_{sep} in (24) is the same as in eqs. (8.10a)-(8.10b) in M&S.

2 The cross-section

On the previous tutorial, we computed some Feynman amplitudes. The final prediction in QFT, however, is not a Feynman amplitude, but rather a cross-section. Let's consider a process in which two particles in the initial state interact and produce two particles in the final state. We have

$$p_i = \left(E_i, \vec{p_i}\right), \quad i = 1, 2, \tag{25a}$$

$$p_j = \left(E_j, \vec{p_j}\right), \quad j = 1, 2. \tag{25b}$$

Let's also suppose that the particles are in a definite polarization or spin state, so we do not have to sum over them. Also, we now consider the process happening within a finite region of space V in a finite amount of time T. This changes the normalizations of the fields. Under these assumptions, the matrix element can be written as,

$$S_{fi} \coloneqq \langle f | \, S \, | i \rangle \simeq \delta_{fi} + (2\pi)^4 \delta \left(\sum_j p_j - \sum_i p_i \right) \Pi_i \sqrt{\frac{1}{2VE_i}} \Pi_j \sqrt{\frac{1}{2VE_j}} \Pi_\ell \sqrt{2m_\ell}\, \mathcal{M}, \tag{26}$$

[1]Imagine in General Relativity, and so on...

where $\mathcal{M}$ is the Feynman amplitude. The $\simeq$ sign is there because the delta function changes in a finite volume and finite time. Namely,

$$(2\pi)^4\delta\left(\sum_j p_j - \sum_i p_i\right) = \lim_{\substack{T\to\infty \\ V\to\infty}} \int_{-T/2}^{T/2} \mathrm{d}t \int_V \mathrm{d}^3x \mathrm{e}^{\mathrm{i}x\left(\sum_f P_f - \sum_i p_i\right)}$$
$$\equiv \lim_{\substack{T\to\infty \\ V\to\infty}} \delta_{TV}\left(\sum_j p_j - \sum_i p_i\right). \quad (27)$$

Therefore, we must replace $(2\pi)^4\delta\left(\sum_j p_j - \sum_i p_i\right)$ with $\delta_{TV}\left(\sum_j p_j - \sum_i p_i\right)$ in our expressions.

Now, the transition probability from $|i\rangle$ to $|f\rangle$ is $|S_{fi}|^2$, as in non-relativistic quantum mechanics. Hence, the rate of transition is equal to

$$w = \frac{|S_{fi}|^2}{T}. \quad (28)$$

At this point, we face a mathematical problem. Since S_{fi} has a δ inside it, its square will have the square of the δ distribution. The product of any two distribution is mathematically ill-defined. How to deal with this problem? First of all, let's notice that we run into it because of an over-simplification. We consider particles with exactly determined momenta, and this is non-physical due to the Heinsenberg uncertainty principle and to the finite resolution of any observation. To get the meaningful cross-section, one should consider wave-packets in the initial and final states, i.e., states with momenta that can range within a finite interval. However, the treatment becomes more complicated if one uses wave-packets, hence we use the exactly known momenta, but we have to pay the price for this, i.e., we need to pursue a non-rigorous treatment where we deal with the (non-existing) quantity "$\delta^2(p)$". We now "compute" the square of the Dirac delta by means of a "trick",

$$\text{"}\delta^2(p)\text{"} = \left(\lim_{\substack{T_1\to\infty \\ V_1\to\infty}} \int_{-T_1/2}^{T_1/2} \frac{\mathrm{d}t_1}{(2\pi)^4} \int_{V_1} \mathrm{d}^3x_1 \mathrm{e}^{\mathrm{i}x_1 p}\right) \cdot$$
$$\cdot \left(\lim_{\substack{T_2\to\infty \\ V_2\to\infty}} \int_{-T_2/2}^{T_2/2} \frac{\mathrm{d}t_2}{(2\pi)^4} \int_{V_2} \mathrm{d}^3x_2 \mathrm{e}^{\mathrm{i}x_2 p}\right)$$
$$= \lim_{\substack{T_1\to\infty \\ V_1\to\infty}} \lim_{\substack{T_2\to\infty \\ V_2\to\infty}} \int_{-T_1/2}^{T_1/2} \mathrm{d}t_1 \int_{-T_2/2}^{T_2/2} \mathrm{d}t_2 \int_{V_1} \mathrm{d}^3x_1 \int_{V_2} \mathrm{d}^3x_2 \frac{\mathrm{e}^{\mathrm{i}p(x_1+x_2)}}{(2\pi)^8}. \quad (29)$$

Now we make the change of variables $x = x_1 + x_2$, $\mathrm{d}x = \mathrm{d}x_2$ in the integral labeled with 2,

$$\text{“}\delta^2(p)\text{”} = \lim_{\substack{T_1\to\infty \\ V_1\to\infty}} \lim_{\substack{T_2\to\infty \\ V_2\to\infty}} \int_{-T_1/2}^{T_1/2} \mathrm{d}t_1 \int_{t_1-T_2/2}^{t_1+T_2/2} \mathrm{d}t \int_{V_1} \mathrm{d}^3x_1 \int_{V_1+V_2} \mathrm{d}^3x \frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^8}. \quad (30)$$

Now we take the limits $T_2, V_2 \to \infty$ first,

$$\begin{aligned} \text{“}\delta^2(p)\text{”} &= \lim_{\substack{T_1\to\infty \\ V_1\to\infty}} \left[\int_{-T_1/2}^{T_1/2} \frac{\mathrm{d}t_1}{(2\pi)^4} \int_{V_1} \mathrm{d}^3x_1 \right] \left[\int_{-\infty}^{\infty} \mathrm{d}t \int_{\text{space}} \mathrm{d}^3x \frac{\mathrm{e}^{\mathrm{i}px}}{(2\pi)^4} \right] \\ &= \lim_{\substack{T_1\to\infty \\ V_1\to\infty}} [T_1V_1] \frac{\delta(p)}{(2\pi)^4}. \end{aligned} \quad (31)$$

At this point, we claim that the physical process happens in the finite time T_1 and in the finite volume V_1, hence we can neglect the limits $T_1, V_1 \to \infty$. In the light of this, in our expression we will use the formula

$$(2\pi)^4 \text{“}\delta^2(p)\text{”} \simeq T_1V_1\, \delta(p), \quad (32)$$

which in terms of δ_{TV} becomes

$$(2\pi)^8 \text{“}\delta^2(p)\text{”} = T_1V_1\delta_{TV}(p). \quad (33)$$

This allows us to write

$$\begin{aligned} w &= \frac{1}{T}(2\pi)^8\delta \left(\sum_j p_j - \sum_i p_i \right)^2 \Pi_i \left(\frac{1}{2VE_i} \right) \Pi_j \left(\frac{1}{2VE_j} \right) \Pi_\ell \left(2m_\ell\right) |\mathcal{M}|^2 \\ &= \frac{TV}{T}(2\pi)^4\delta \left(\sum_j p_j - \sum_i p_i \right) \Pi_i \left(\frac{1}{2VE_i} \right) \Pi_j \left(\frac{1}{2VE_j} \right) \Pi_\ell \left(2m_\ell\right) |\mathcal{M}|^2 \\ &= V(2\pi)^4\delta \left(\sum_j p_j - \sum_i p_i \right) \Pi_i \left(\frac{1}{2VE_i} \right) \Pi_j \left(\frac{1}{2VE_j} \right) \Pi_\ell \left(2m_\ell\right) |\mathcal{M}|^2. \end{aligned} \quad (34)$$

This is the transition rate to a final state with exact final momenta. Now, we know that we will always have some uncertainty in measuring the momenta, so it makes sense to consider momenta in the interval $(p_f, p_f + \mathrm{d}p_f)$. We know that, in a discretized system (finite volume), the density of states is $\dfrac{V}{(2\pi)^3}\mathrm{d}^3p_f$. The “differential cross section” is the transition rate into this group of final states, per unit incident flux of particles in the initial state. The incident flux is $\phi = nv_{\text{sep}} = \dfrac{v_{\text{sep}}}{V}$, where v_{sep} is the separation velocity of the colliding particles. We have,

$$\mathrm{d}\sigma := \frac{w}{\phi} \Pi_j \frac{V\mathrm{d}^3p_j}{(2\pi)^3}$$

$$= (2\pi)^4 \delta\left(\sum_f p_f - \sum_i p_i\right) \frac{1}{4E_1E_2 v_{\text{sep}}} \Pi_\ell \left(2m_\ell\right) \Pi_j \left(\frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_j}\right) |\mathcal{M}|^2$$
$$= \frac{1}{64\pi^2 v_{\text{sep}} E_1 E_2 E_1' E_2'} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2 \delta\left(p_1' + p_2' - p_1 - p_2\right) \mathrm{d}^3 p_1' \mathrm{d}^3 p_2', \tag{35}$$

if the particles move collinearly. In this case it holds

$$E_1 E_2 v_{\text{sep}} = \left[(p_1 p_2)^2 - m_1^2 m_2^2\right]^{1/2}. \tag{36}$$

Two important frames in which the particles move collinearly are the center of mass frame (COM) and the laboratory frame (LAB).

The COM frame is defined by $\vec{p}_1 = -\vec{p}_2$ and so

$$v_{\text{sep}} = \frac{|\vec{p}_1|}{E_1} + \frac{|\vec{p}_2|}{E_2} = |\vec{p}_1| \frac{E_1 + E_2}{E_1 E_2}. \tag{37}$$

In the LAB frame, one particle is at rest, so $\vec{p}_2 = 0$ and

$$v_{\text{sep}} = \frac{|\vec{p}_1|}{E_1}. \tag{38}$$

The last step to make the differential cross-section observable is to remove the last delta, which is not observable. We then need to integrate over the final momenta, because they can be arbitrary as long as they respect energy–momentum conservation (guaranteed by the delta). We perform the integration over $\vec{p_2'}$ first,

$$\mathrm{d}\sigma = \mathrm{d}^3 p_1' \int \mathrm{d}^3 p_2' \frac{1}{64\pi^2 v_{\text{sep}} E_1 E_2 E_1' E_2'} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2 \delta\left(p_1' + p_2' - p_1 - p_2\right)$$
$$= \frac{1}{64\pi^2 v_{\text{sep}} E_1 E_2 E_1' E_2'} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2 \delta\left(E_1' + E_2' - E_1 - E_2\right) |\vec{p_1'}|^2 \mathrm{d}p_1' \mathrm{d}\Omega', \tag{39}$$

where in the last step we used

$$\mathrm{d}^3 p = |\vec{p}|^2 \mathrm{d}|\vec{p}| \mathrm{d}\Omega = |\vec{p}| E \mathrm{d}E \mathrm{d}\Omega = |\vec{p}| E \mathrm{d}E \sin(\theta) \mathrm{d}\theta \mathrm{d}\phi, \tag{40}$$

which corresponds to going to spherical coordinates in the momentum space. We now integrate over $\vec{p_1'}$. This will kill the last delta and leave us with an observable quantity. We use the formula

$$\int f(x,y)\delta[g(x,y)]\mathrm{d}x = \int f(x,y)\delta[g(x,y)] \left(\frac{\partial x}{\partial g}\right)\bigg|_y \mathrm{d}g = \left.\frac{f(x,y)}{\left(\frac{\partial g}{\partial x}\right)\big|_y}\right|_{g=0} \tag{41}$$

to get,

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 v_{\text{sep}} E_1 E_2 E_1' E_2'} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2 |\vec{p}_1'|^2 \mathrm{d}\Omega' \left[\frac{\partial \left(E_1' + E_2'\right)}{\partial |p_1'|}\right]^{-1}. \tag{42}$$

Remind that $\vec{p'}_2 = \vec{p}_1 + \vec{p}_2 - \vec{p'}_1$ from energy–momentum conservation. In the COM frame, we have

$$E'^2 = m^2 + |\vec{p'}|^2 \Longrightarrow \frac{\partial \left(E_1' + E_2'\right)}{\partial |p_1'|} = |p_1'| \frac{E_1 + E_2}{E_1' E_2'}. \tag{43}$$

Hence, in the COM we obtain

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'}\right|_{\text{COM}} = \frac{1}{64\pi^2 (E_1 + E_2)^2} \frac{|p_1'|}{|p_1|} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2. \tag{44}$$

This is the observable quantity that can be compared with the scattering experiments.

Now we compute the differential cross-section in the LAB. One particle is stationary, so

$$\vec{p'}_2 = 0, \quad p_2 = E_2 = m_2. \tag{45}$$

We start again from

$$\mathrm{d}\sigma = f(p_1', p_2') |\vec{p'}_1|^2 \mathrm{d}\Omega' \left[\frac{\partial \left(E_1' + E_2'\right)}{\partial |p_1'|}\right]^{-1}, \tag{46}$$

with $f(p_1', p_2') = \dfrac{1}{64\pi^2 v_{\text{sep}} E_1 E_2 E_1' E_2'} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2$. The relative velocity in the lab frame is $v_{\text{sep}} = \dfrac{|\vec{p}_1|}{E_1}$. Substituting into $\mathrm{d}\sigma$ we get

$$\begin{aligned}\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'}\right|_{\text{LAB}} &= \frac{1}{64\pi^2 \frac{|p_1|}{E_1} E_1 E_2 E_1' E_2'} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2 |\vec{p'}_1|^2 \mathrm{d}\Omega' \left[\frac{\partial \left(E_1' + E_2'\right)}{\partial |p_1'|}\right]^{-1} \\ &= \frac{1}{64\pi^2 E_1' E_2'} \frac{|\vec{p'}_1|^2}{|p_1| m_2} \left(\Pi_\ell 2m_\ell\right) |\mathcal{M}|^2 \mathrm{d}\Omega' \left[\frac{\partial \left(E_1' + E_2'\right)}{\partial |p_1'|}\right]^{-1}. \end{aligned} \tag{47}$$

Now we compute the partial derivative. We know that

$$E_1'^2 = |\vec{p'}_1|^2 + m_1^2, \quad E_2'^2 = |\vec{p'}_2|^2 + m_2^2. \tag{48}$$

Conservation of four-momentum gives us $p_1 + p_2 = p_1' + p_2'$, which implies

$$\vec{p'}_2 = \vec{p}_1 - \vec{p'}_1 \Longrightarrow \left|\vec{p'}_2\right|^2 = |\vec{p}_1|^2 + \left|\vec{p'}_1\right|^2 - 2\,|\vec{p}_1|\,\left|\vec{p'}_1\right| \cos(\theta'). \tag{49}$$

It follows

$$E_1' = \sqrt{m_1^2 + \left|\vec{p'}_1\right|^2}, \quad E_2' = \sqrt{m_2^2 + |\vec{p}_1|^2 + \left|\vec{p'}_1\right|^2 - 2\,|\vec{p}_1|\left|\vec{p'}_1\right|\cos(\theta')}. \tag{50}$$

Now we can compute the partial derivatives

$$\frac{\partial E_1'}{\partial |p_1'|} = \frac{1}{2}\left(m_1^2 + \left|\vec{p'}_1\right|^2\right)^{-1/2} 2\left|\vec{p'}_1\right| = \frac{\left|\vec{p'}_1\right|}{\left(m_1^2 + \left|\vec{p'}_1\right|^2\right)^{1/2}} = \frac{\left|\vec{p'}_1\right|}{E_1'}, \tag{51a}$$

$$\begin{aligned}\frac{\partial E_1'}{\partial |p_1'|} &= \frac{1}{2}\left(m_2^2 + |\vec{p}_1|^2 + \left|\vec{p'}_1\right|^2 - 2\,|\vec{p}_1|\left|\vec{p'}_1\right|\cos(\theta')\right)^{-1/2}\left(2\left|\vec{p'}_1\right| - 2\,|\vec{p}_1|\cos(\theta')\right)\\ &= \frac{\left|\vec{p'}_1\right| - |\vec{p}_1|\cos(\theta')}{E_2'}.\end{aligned} \tag{51b}$$

Hence we get,

$$\begin{aligned}\frac{\partial\left(E_1' + E_2'\right)}{\partial |p_1'|} &= \frac{\left|\vec{p'}_1\right|}{E_1'} + \frac{\left|\vec{p'}_1\right| - |\vec{p}_1|\cos(\theta')}{E_2'} = \frac{\left(E_1' + E_2'\right)\left|\vec{p'}_1\right| - E_1'\,|\vec{p}_1|\cos(\theta')}{E_1'E_2'}\\ &= \frac{\left(E_1 + E_2\right)\left|\vec{p'}_1\right| - E_1'\,|\vec{p}_1|\cos(\theta')}{E_1'E_2'} = \frac{\left(E_1 + m_2\right)\left|\vec{p'}_1\right| - E_1'\,|\vec{p}_1|\cos(\theta')}{E_1'E_2'}.\end{aligned} \tag{52}$$

The cross-section in the LAB is then

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega'}\right|_{\mathrm{LAB}} = \frac{1}{64\pi^2 m_2}\frac{1}{\left(E_1 + m_2\right)\left|\vec{p'}_1\right| - E_1'\,|\vec{p}_1|\cos(\theta')}\frac{\left|\vec{p'}_1\right|^2}{|\vec{p}_1|}\left(\Pi_\ell 2m_\ell\right)|\mathcal{M}|^2. \tag{53}$$

3 The spin-sums lemma

The spin-sums lemma, or Casimir's trick, is the following statement,

$$\sum_{r,r'}\left(\overline{u}_{r'}\left(\vec{p'}\right)Av_r\left(\vec{p}\right)\right)\left(\overline{u}_{r'}\left(\vec{p'}\right)Bv_r\left(\vec{p}\right)\right)^\dagger = \frac{1}{4m^2}\,\mathrm{Tr}\left(\left(\not{p}' + m\right)A\left(\not{p} - m\right)\tilde{B}\right), \tag{54}$$

where A and B are matrices built out of γ matrices, and $\tilde{B} = \gamma^0 B^\dagger \gamma^0$.

The proof follows,

$$\sum_{r,r'}\left(\overline{u}_{r'}\left(\vec{p'}\right)Av_r\left(\vec{p}\right)\right)\left(\overline{u}_{r'}\left(\vec{p'}\right)Bv_r\left(\vec{p}\right)\right)^\dagger$$

$$
\begin{aligned}
&= \sum_{r,r'} \left(u^\dagger_{r'} (\vec{p'})\, \gamma^0 A v_r (\vec{p}) \right) \left(u^\dagger_{r'} (\vec{p'})\, \gamma^0 B v_r (\vec{p}) \right)^\dagger \\
&= \sum_{r,r'} \left(u^\dagger_{r'} (\vec{p'})\, \gamma^0 A v_r (\vec{p}) \right) \left(v^\dagger_{r} (\vec{p})\, B^\dagger {\gamma^0}^\dagger u_{r'} (\vec{p'}) \right) \\
&= \sum_{r,r'} \left(u^\dagger_{r'} (\vec{p'})\, \gamma^0 A v_r (\vec{p}) \right) \left(v^\dagger_{r} (\vec{p})\, \gamma^0 \gamma^0 B^\dagger \gamma^0 u_{r'} (\vec{p'}) \right) \\
&= \sum_{r,r'} \left(\overline{u}_{r'} (\vec{p'})\, A v_r (\vec{p}) \right) \left(\overline{v}_{r} (\vec{p})\, \gamma^0 B^\dagger \gamma^0 u_{r'} (\vec{p'}) \right) .
\end{aligned}
\tag{55}
$$

Now we rewrite the same object in spinorial index notation (so far we have been using the matrix notation),

$$
\begin{aligned}
&\sum_{r,r'} \left(\overline{u}_{r'} (\vec{p'})_I\, A_{IJ} v_r (\vec{p})_J \right) \left(\overline{v}_r (\vec{p})_K\, \gamma^0_{KL} B^\dagger_{LM} \gamma^0_{MN} u_{r'} (\vec{p'})_N \right) \\
&= \sum_{r'} \left(u_{r'} (\vec{p'})_N\, \overline{u}_{r'} (\vec{p'})_I \right) A_{IJ} \sum_r \left(v_r (\vec{p})_J\, \overline{v}_r (\vec{p})_K \right) \left(\gamma^0_{KL} B^\dagger_{LM} \gamma^0_{MN} \right) \\
&= \left(\frac{\not{p}' + m}{2m} \right)_{NI} A_{IJ} \left(\frac{\not{p} - m}{2m} \right)_{JK} \left(\gamma^0_{KL} B^\dagger_{LM} \gamma^0_{MN} \right) \\
&= \mathrm{Tr} \left(\frac{\not{p}' + m}{2m} A \frac{\not{p} - m}{2m} \gamma^0 B^\dagger \gamma^0 \right) \\
&= \frac{1}{4m^2} \mathrm{Tr} \left(\left(\not{p}' + m \right) A \left(\not{p} - m \right) \gamma^0 B^\dagger \gamma^0 \right) \\
&= \frac{1}{4m^2} \mathrm{Tr} \left(\left(\not{p}' + m \right) A \left(\not{p} - m \right) \tilde{B} \right) .
\end{aligned}
\tag{56}
$$

In the second equality in (56), we used the expressions for the projectors onto the positive and negative energy states for the Dirac spinors,[2]

$$
\Lambda^+ (\vec{p}) = \sum_r u_r (\vec{p})\, \overline{u}_r (\vec{p}) = \frac{\not{p} + m}{2m}, \tag{57a}
$$

$$
\Lambda^- (\vec{p}) = -\sum_r v_r (\vec{p})\, \overline{v}_r (\vec{p}) = -\frac{\not{p} - m}{2m}. \tag{57b}
$$

4 $e^- e^+$ production in electromagnetic field

We consider the following initial and final states

$$
|i\rangle = |0\rangle, \quad |f\rangle = c^\dagger_t (\vec{p_1})\, d^\dagger_s (\vec{p_2})\, |0\rangle, \tag{58}
$$

and an electromagnetic field of the form

$$
A_\mu = \left(0, 0, a\mathrm{e}^{-\mathrm{i}\omega t}, 0 \right), \tag{59a}
$$

[2] Did you prove these formulas after the tutorial on the Dirac equation?

$$A^\mu = \eta^{\mu\nu} A_\nu = \left(0, 0, \eta^{22} A_2, 0\right) = \left(0, 0, -a\mathrm{e}^{-\mathrm{i}\omega t}, 0\right). \tag{59b}$$

We consider the first-order S-matrix in QED,

$$\begin{aligned}
\langle f| S^{(1)} |i\rangle &= \mathrm{i}e \int \mathrm{d}^4 x \overline{\psi}^- (x) \not{A}(x) \psi^-(x) \\
&= \frac{\mathrm{i}e}{(2\pi)^3} \int \mathrm{d}^4 x \sum_{r' s'} \int \mathrm{d}^3 q_1 \mathrm{d}^3 q_2 \sqrt{\frac{m}{E_{\vec{q_1}}}} \sqrt{\frac{m}{E_{\vec{q_2}}}} \cdot \\
&\cdot \langle 0| c_r (\vec{p_1}) \, d_s (\vec{p_2}) \left[c^\dagger_{r'} (\vec{q_1}) \, \overline{u}_{r'} (\vec{q_1}) \, \mathrm{e}^{\mathrm{i} q_1 x}\right] \left(-\gamma_2 a \mathrm{e}^{-\mathrm{i}\omega t}\right) \left[d^\dagger_{s'} (\vec{q_2}) \, v_{s'} (\vec{q_2}) \, \mathrm{e}^{\mathrm{i} q_2 x}\right] |0\rangle \\
&= -\frac{\mathrm{i}ea}{(2\pi)^3} \sqrt{\frac{m}{E_1}} \sqrt{\frac{m}{E_2}} \int \mathrm{d}^4 x \overline{u}_r (\vec{p_1}) \, \gamma_2 v_s (\vec{p_2}) \, \mathrm{e}^{-\mathrm{i} p_1 x + \mathrm{i} p_2 x - \mathrm{i}\omega t} \\
&= -\frac{\mathrm{i}eam}{(2\pi)^3} \frac{1}{\sqrt{E_1 E_2}} (2\pi)^4 \delta(\vec{p_2} - \vec{p_1}) \delta\left(E_2 + E_1 - \omega\right) \overline{u}_r (\vec{p_1}) \, \gamma_2 v_s (\vec{p_2}) \, .
\end{aligned} \tag{60}$$

The Feynman amplitude is

$$\mathcal{M} = -\mathrm{i}ea \overline{u}_r (\vec{p_1}) \, \gamma_2 v_s (\vec{p_2}) \, . \tag{61}$$

We are not assuming any definite polarization for the particles in the final state, so we must sum over the polarizations of the final state

$$\begin{aligned}
|\mathcal{M}|^2 &= -\mathrm{i}\,\mathrm{i}(ea)^2 \sum_{rs} \overline{u}_r (\vec{p_1}) \, \gamma_2 v_s (\vec{p_2}) \, \overline{v}_s (\vec{p_2}) \, \gamma_2 u_r (\vec{p_1}) \\
&= e^2 a^2 \, \mathrm{Tr} \left(\frac{\not{p}_2 - m}{2m} \gamma_2 \frac{\not{p}_1 + m}{2m} \gamma^2\right) = \frac{e^2 a^2}{4m^2} \left[\mathrm{Tr}\left(\not{p}_2 \gamma_2 \not{p}_1 \gamma^2\right) - m^2 \, \mathrm{Tr}\left(\gamma_2 \gamma^2\right)\right].
\end{aligned} \tag{62}$$

Here we use the “Casimir’s trick”, proved in the next section. We now make use of the following relations

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta\right) &= \eta^{\alpha\beta}, \quad \mathrm{Tr}\left(\text{odd } \# \text{ of } \gamma\right) = 0, \\
\mathrm{Tr}\left(\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta\right) &= 4\left(\eta^{\alpha\beta}\eta^{\gamma\delta} - \eta^{\alpha\gamma}\eta^{\beta\delta} + \eta^{\alpha\delta}\eta^{\gamma\beta}\right).
\end{aligned} \tag{63}$$

We get,

$$\begin{aligned}
\mathrm{Tr}\left(\not{p}_2 \gamma_2 \not{p}_1 \gamma^2\right) &= 4\left(p_{2\mu} p_{1\nu}\right) \left(\eta^\mu{}_2 \eta^{\nu 2} - \eta^{\mu\nu} \eta^2{}_2 + \eta^{\mu 2} \eta^\nu{}_2\right) \\
&= 4\left[(p_2)_2 (p_1)^2 - (p_2)_\mu (p_1)_\nu \eta^{\mu\nu} + (p_1)_2 (p_2)^2\right] \\
&= -4\left[(p_2)_2 (p_1)_2 + (p_2)_\mu (p_1)^\mu + (p_1)_2 (p_2)_2\right].
\end{aligned} \tag{64}$$

Explicitly, the momenta are given by

$$(p_1)^\mu = \left(E_1, |\vec{p_1}| \sin(\theta) \cos(\phi), |\vec{p_1}| \sin(\theta) \sin(\phi), |\vec{p_1}| \cos(\theta)\right), \tag{65a}$$

$$(p_2)^\mu = \left(E_2, -|\vec{p_2}| \sin(\theta) \cos(\phi), -|\vec{p_2}| \sin(\theta) \sin(\phi), -|\vec{p_2}| \cos(\theta)\right). \tag{65b}$$

It follows,

$$\mathrm{Tr}\left(\not{p}_2\gamma_2\not{p}_1\gamma^2\right) = 4\left[-2|\vec{p_2}||\vec{p_1}|\sin(\theta)^2\sin(\phi)^2 + E_1E_2 - \vec{p_1}\cdot\vec{p_2}\right]. \tag{66}$$

Also, $\mathrm{Tr}\left(\gamma_2\gamma^2\right) = 4\eta^{22} = 4$. The Feynman amplitude squared becomes,

$$|\mathcal{M}|^2 = \frac{e^2a^2}{m^2}\left[E_1E_2 + |\vec{p_1}||\vec{p_2}| + m^2 - 2|\vec{p_2}||\vec{p_1}|\sin(\theta)^2\sin(\phi)^2\right]. \tag{67}$$

This concludes the computation of the amplitude. We now turn to the computation of the differential cross-section $\mathrm{d}\sigma = wV\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6}$, with $w = \frac{|S_{fi}|^2}{T}$. Now let's consider the part of S_{fi} which is not in $\mathcal{M}$ and call it R.

$$R = \frac{1}{(2\pi)^3}\frac{m}{\sqrt{E_1E_2}}\frac{(2\pi)^4}{(2\pi)^6}\delta(\vec{p_2}-\vec{p_1})\delta(E_2+E_1-\omega), \tag{68}$$

which implies

$$|R|^2 = \frac{m^2}{V^2E_1E_2}TV\frac{(2\pi)^8}{(2\pi)^6(2\pi)^4}\delta(\vec{p_2}-\vec{p_1})\delta(E_2+E_1-\omega), \tag{69}$$

where $(2\pi)^3 \to V$ because of the finite limit assumption. Then we have,

$$|S_{fi}|^2 = |\mathcal{M}|^2|R|^2, \tag{70}$$

and

$$\begin{aligned}\mathrm{d}\sigma = &\frac{e^2a^2}{E_1E_2}\frac{1}{(2\pi)^2}\delta(\vec{p_2}-\vec{p_1})\delta(E_2+E_1-\omega)\cdot\\ &\cdot\left[E_1E_2 + |\vec{p_1}||\vec{p_2}| + m^2 - 2|\vec{p_2}||\vec{p_1}|\sin(\theta)^2\sin(\phi)^2\right]\mathrm{d}^3p_1\mathrm{d}^3p_2.\end{aligned} \tag{71}$$

We now need to integrate over $\vec{p_2}$,

$$\begin{aligned}\mathrm{d}\sigma &= \frac{e^2a^2}{(2\pi)^2}\frac{\delta(2E_1-\omega)}{E_1^2}\left[E_1^2 + |\vec{p_1}| + m^2 - 2|\vec{p_1}|^2\sin(\theta)^2\sin(\phi)^2\right]\mathrm{d}^3p_1\\ &= \frac{e^2a^2}{2E_1^2(2\pi)^2}\delta(E_1-\omega/2)\left[2E_1^2 - 2(E_1^2-m^2)\sin(\theta)^2\sin(\phi)^2\right]\mathrm{d}^3p_1,\end{aligned} \tag{72}$$

where we used $\delta(\alpha x) = \delta(x)/|\alpha|$. Now we integrate over $\vec{p_1}$ by using

$$E = \sqrt{|\vec{p}|^2+m^2} \Longrightarrow \mathrm{d}E = \frac{2|\vec{p}|\mathrm{d}|\vec{p}|}{2\sqrt{|\vec{p}|^2+m^2}} \Longrightarrow E\mathrm{d}E = |\vec{p}|\mathrm{d}|\vec{p}|, \tag{73a}$$

$$\mathrm{d}^3p = |\vec{p}|^2\mathrm{d}|\vec{p}|\mathrm{d}\Omega = |\vec{p}|E\mathrm{d}E\mathrm{d}\Omega = |\vec{p}|E\mathrm{d}E\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi. \tag{73b}$$

The cross-section then is,

$$\sigma = \frac{e^2a^2}{(2\pi)^2}\int\mathrm{d}E_1\int\mathrm{d}\Omega\frac{|\vec{p_1}|}{2E_1}\delta(E_1-\omega/2)\left[2E_1^2 - 2(E_1^2-m^2)\sin(\theta)^2\sin(\phi)^2\right]$$

$$= \frac{e^2a^2}{(2\pi)^2}\int_0^\pi \mathrm{d}\theta\sin(\theta)\int_0^{2\pi}\mathrm{d}\phi\frac{\sqrt{\omega^2/4-m^2}}{\omega}\left[\omega^2/2-2(\omega^2/4-m^2)\sin(\theta)^2\sin(\phi)^2\right]. \tag{74}$$

Since

$$\int_0^\pi \sin(\theta)^3\mathrm{d}\theta = 4/3, \tag{75a}$$

$$\int_0^{2\pi} \sin(\phi)^2\mathrm{d}\phi = \pi, \tag{75b}$$

$$\int_0^\pi \sin(\theta)\mathrm{d}\theta = 2, \tag{75c}$$

we get

$$\begin{aligned}\sigma &= \frac{e^2a^2}{(2\pi)^2}\frac{\sqrt{\omega^2/4-m^2}}{\omega}\left[\pi\omega^2-\frac{4}{3}\pi\left(\omega^2/4-m^2\right)\right]\\ &= \frac{e^2a^2}{3\pi}\frac{\sqrt{\omega^2/4-m^2}}{\omega}\left[\omega^2+2m^2\right].\end{aligned} \tag{76}$$

This is meaningful only if,

$$\frac{\omega^2}{4}\geqslant m^2 \Longrightarrow \omega\geqslant 2m \xrightarrow{\text{SI units}} \hbar\omega\geqslant 2m. \tag{77}$$

The process can happen only if the energy provided by the external electromagnetic field is larger than or equal to the sum of the masses of the final particles. However, for the equality σ is defined, but it is zero.

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