Tutorial 13

FK8027 - Quantum Field Theory

Monday 11$^{\text{th}}$ February, 2019

Topics for today

- Propagating degrees of freedom for spin-1 fields
- Weak SU(2) isospin charges

1 Propagating degrees of freedom for spin-1 fields

Massive spin-1 field. In the previous tutorial we derived the Proca equation for a massive spin-1 field starting from the Lagrangian density

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_W^2 W^\mu W_\mu \tag{1}$$

with $F^{\mu\nu} := \partial^\mu W^\nu - \partial^\nu W^\mu$, analogously to the Maxwell tensor for the electromagnetic field. The Proca equation results in

$$\Box W^\mu + m_W^2 W^\mu = 0, \tag{2}$$

with the Lorenz *condition*

$$\partial_\mu W^\mu = 0. \tag{3}$$

Note that this condition is not a choice that we make, contrary to the case of a massless spin-1 field. Rather, it is a *necessary condition* following from the field equations themselves, as we saw in the previous tutorial.

The vector W^μ contains four components, and through the Lorenz condition we can express one in terms of the others. This tells us that only three components are independent. This has to be the case, because a spin-1 particle in Minkowski spacetime must have three degrees of freedom, associated to the three possible values of the spin quantum number $S_z = -1, 0, 1$.

Massless spin-1 field. In the massless spin-1 case, the equations of motion do not imply any condition; the explicit computation in the previous tutorial shows that the Lorenz condition originates from the mass term. On the other hand, the action and the field equations are gauge invariant and so we are free to *choose* a gauge. The choice of the gauge, as we shall see explicitly, eliminates two degrees of freedom, leaving us with only two propagating degrees of freedom. This is the correct number because a massless spin-1 field in Minkowski spacetime must have only the polarizations $S_z = -1, 1$.[1]

[1] Actually, a massless particle with any spin n in Minkowski spacetime must have only the two degrees of freedom $S_z = -n, n$.

Consider the Maxwell equations

$$\partial_\nu F^{\mu\nu} = J^\mu. \tag{4}$$

In the Lorenz *gauge*, these equations reduce to

$$\Box A^\mu = J^\mu, \quad \partial_\mu A^\mu = 0. \tag{5}$$

As we know, a solution to these equations is

$$A^\mu(x) = A_0^\mu(x) + \int \mathrm{d}^4x' G(x-x') J^\mu(x'), \tag{6}$$

where $A_0^\mu(x)$ is a solution to the homogeneous wave equations and

$$G(x-x') = \frac{1}{(2\pi)^4} \int d^4p \frac{\mathrm{e}^{-\mathrm{i}\,p\,(x-x')}}{p^2 + \mathrm{i}\epsilon} \tag{7}$$

is the Green's function for the wave equation. A solution to the homogeneous equation is

$$A_0^\mu(x) = N\epsilon^\mu \mathrm{e}^{-\mathrm{i}kx}, \tag{8}$$

provided that $k_\mu k^\mu = 0$, with N normalization constant. The Lorenz gauge tells us that

$$\begin{aligned} 0 = \partial_\mu A^\mu(x) &= \partial_\mu A_0^\mu(x) + \int \mathrm{d}^4x' \partial_\mu G(x-x') J^\mu(x') \\ &= \partial_\mu A_0^\mu(x) - \int \mathrm{d}^4x' \partial_{\mu'} G(x-x') J^\mu(x') \\ &= \partial_\mu A_0^\mu(x) + \int \mathrm{d}^4x' G(x-x') \partial_{\mu'} J^\mu(x') \\ &= \partial_\mu A_0^\mu(x) = N\epsilon^\mu \partial_\mu \mathrm{e}^{\mathrm{i}kx} = \mathrm{i}N\epsilon^\mu k_\mu \mathrm{e}^{\mathrm{i}kx} \Longrightarrow e^\mu k_\mu = 0, \end{aligned} \tag{9}$$

where we have used $\partial_\mu G(x-x') = -\partial_{\mu'} G(x-x')$ due to the form of the Green function, and we have integrated by parts. The boundary term is zero because we assume that the current J^μ vanishes at infinity, and the remaining term is zero because $\partial_{\mu'} J^\mu(x') = 0$, due to $\partial_\mu \partial_\nu F^{\nu\mu} \equiv 0$ ($\partial_\mu \partial_\nu$ is symmetric, whereas $F^{\nu\mu}$ is antisymmetric).

Now, the Lorenz gauge does not fix the gauge completely, because the Maxwell equations (5) are still invariant under the following "residual" gauge transformation,

$$A^\mu \to A^\mu + \partial^\mu \lambda, \quad \Box\lambda = \partial_\mu \partial^\mu \lambda = 0. \tag{10}$$

This can be easily seen by applying the transformation,

$$\Box A^\mu \to \Box\left(A^\mu + \partial^\mu \lambda\right) = \Box A^\mu + \partial^\mu \Box \lambda = \Box A^\mu, \tag{11}$$

where we have used the commutation between partial derivatives. Hence, solving (10), we have

$$\lambda = N_1 \mathrm{e}^{-\mathrm{i}kx}, \tag{12}$$

provided that $k_\mu k^\mu = 0$ again, with N_1 free constant parameter (this is a gauge function, so we are not obliged to normalize; in other words, N_1 is completely free). We can choose to have the same momentum k of the solution (8) because we are free to make the gauge choice we like the most.

Then, the residual gauge freedom is

$$A^\mu \to A^\mu + \partial^\mu \lambda = N \epsilon^\mu \mathrm{e}^{-\mathrm{i}kx} - \mathrm{i}\, k^\mu N_1 \mathrm{e}^{-\mathrm{i}kx} + \int \mathrm{d}^4 x' G(x - x') J^\mu(x'), \tag{13}$$

or equivalently

$$\epsilon^\mu \to \epsilon^\mu - \mathrm{i}\frac{N_1}{N} k^\mu. \tag{14}$$

This does not surprise, since $k^\mu k_\mu = 0$ and then the term linear in k^μ does not spoil the Lorenz gauge,

$$0 = \epsilon^\mu k_\mu - \mathrm{i}\frac{N_1}{N} k^\mu k_\mu = \epsilon^\mu k_\mu. \tag{15}$$

We can then use the freedom to choose N_1 in (14) to set to zero another component of ϵ^μ. We then have only two independent components for the polarization vector.

Commonly, one chooses the timelike component to be zero, and the reason is the following. The gauge transformation (14) tells us that the component of ϵ^μ parallel to k^μ is not gauge-invariant—it does change according to (14) itself. The orthogonal component is instead gauge-invariant. Now, the momentum of the photon is

$$k^\mu = (|\vec{k}|, \vec{k}), \tag{16}$$

therefore the gauge-dependent components of ϵ^μ are the timelike one and the one parallel to $\vec{k}$ (the longitudinal). We can fix the residual gauge by setting the timelike component ϵ^0 to zero, choosing a residual gauge transformation with the appropriate value of N_1 in (14), such that

$$\epsilon^0 \to \epsilon^0 - \mathrm{i}\frac{N_1}{N} k^0 = 0 \quad \Longrightarrow \quad N_1 = \frac{N\,\epsilon^0}{\mathrm{i}\,k^0}. \tag{17}$$

After having made this choice, the Lorenz gauge (9) becomes

$$0 = \epsilon_\mu k^\mu = \epsilon_0 k^0\,(= 0) + \epsilon_i k^i = \epsilon_i k^i = -\vec{\epsilon} \cdot \vec{k}, \tag{18}$$

that is, the longitudinal component of ϵ^μ is also zero. We are left with the two transverse components only, which are gauge-independent and therefore physical. They are the two propagating degrees of freedom, corresponding to $S_z = -1, 1$.

We can show why the transverse helicity states of the photon are $S_z = -1, 1$, by performing a rotation in the transverse plane. If we choose a reference frame where the z axis is the longitudinal direction (we can always do that without losing generality), this rotation is parametrized by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

We can consider only the non-trivial 2-dimensional part of it. The 2-dimensional transverse part of the polarization vector transforms as

$$\begin{pmatrix} \epsilon'_1 \\ \epsilon'_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta)\,\epsilon_1 + \sin(\theta)\,\epsilon_2 \\ -\sin(\theta)\,\epsilon_1 + \cos(\theta)\,\epsilon_2 \end{pmatrix}. \tag{20}$$

We can now define the two complex helicity states as [Wei72, p. 255]

$$\epsilon_\pm \coloneqq \epsilon_1 \mp \mathrm{i}\epsilon_2. \tag{21}$$

Under the rotation, they transform as (please check)

$$\epsilon'_\pm = \mathrm{e}^{\pm\mathrm{i}\theta}\,\epsilon_\pm, \tag{22}$$

which is the definition of plane waves having elicities ± 1.

The whole analysis gets much simpler if one chooses the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = \partial_i A^i = 0$, rather than the Lorenz gauge. The Coulomb gauge removes one longitudinal degree of freedom, and renders the equation for A^0 nondynamical (please check). However, the Coulomb gauge is not Lorentz invariant (it only involves the spatial components of A^μ), and therefore one should also prove that the results hold in every inertial frame of reference.

Note that the source terms are not involved at all in the arguments above. I included them to explicitly show that, indeed, they do not contribute.

2 Weak SU(2) isospin charges

We assign the weak isospin charges to the leptonic states in an analogous way as we assign the electromagnetic charges to them. We remind the latter case first.

The electromagnetic charge. The electromagnetic current of the lepton ℓ is

$$J^{\mu}_{\rm EM} = -e\,\bar{\psi}_\ell\,\gamma^\mu\,\psi_\ell = -e\left(\bar{\psi}^{\rm L}_\ell\,\gamma^\mu\,\psi^{\rm L}_\ell + \bar{\psi}^{\rm R}_\ell\,\gamma^\mu\,\psi^{\rm R}_\ell\right). \tag{23}$$

The continuity equation (which follows from $\partial_\mu\partial_\nu F^{\mu\nu} = 0$)

$$\partial_\mu J^\mu = 0 \tag{24}$$

tells us that

$$\partial_0 \int d^3x\,J^0 = -\int d^3x\,\partial_i J^i = -\int_\partial d^2x\,n_i\,J^i = 0, \tag{25}$$

since we assume the current J^μ to vanish at the boundary ∂ of our region of interest. Therefore, we have the conserved charge

$$Q_{\rm EM} = \int d^3x\,J^0 = -e\int d^3x\,\bar{\psi}_\ell\,\gamma^0\,\psi_\ell = -e\int d^3x\,\psi^\dagger_\ell\,\psi_\ell. \tag{26}$$

Inserting the Dirac fields for all the leptons in the theory and simplifying, we get

$$Q_{\rm EM} = -e\sum_\ell\sum_{r=1}^{2}\int d^3p\left[c^\dagger_{r,\ell}(\vec{p})\,c_{r,\ell}(\vec{p}) - d^\dagger_{r,\ell}(\vec{p})\,d_{r,\ell}(\vec{p})\right]. \tag{27}$$

Now recall that the creation and annihilation operators act on the vacuum state as follows,

$$\begin{aligned} c^\dagger_{1,\ell}(\vec{p})\,|0\rangle &= \left|\ell^-, R\right\rangle, & c^\dagger_{2,\ell}(\vec{p})\,|0\rangle &= \left|\ell^-, L\right\rangle, \\ d^\dagger_{1,\ell}(\vec{p})\,|0\rangle &= \left|\ell^+, R\right\rangle, & d^\dagger_{2,\ell}(\vec{p})\,|0\rangle &= \left|\ell^+, L\right\rangle. \end{aligned}$$

Therefore the charge operator acts as

$$\begin{aligned} Q_{\rm EM}\left|\ell^-, R\right\rangle &= -e\left|\ell^-, R\right\rangle, & Q_{\rm EM}\left|\ell^-, L\right\rangle &= -e\left|\ell^-, L\right\rangle, \\ Q_{\rm EM}\left|\ell^+, R\right\rangle &= +e\left|\ell^+, R\right\rangle, & Q_{\rm EM}\left|\ell^+, L\right\rangle &= +e\left|\ell^+, L\right\rangle, \end{aligned}$$

so it returns the electromagnetic charge of the state.

Note that the relative minus sign in (27) is crucial to obtain the correct values of the charges.

The weak isospin charges. The isospin is a quantum number analogous to the spin. They both characterize a quantum state under a SU(2) rotation. However, the spin concerns rotations in spacetime, whereas the isospin concerns abstract (or "internal") rotations of the leptonic fields between themselves. For this reason, the structure of the isospin charge operators $I^{\rm W}_i$ is the same as the spin ones S_i.

We define the currents [eq. (17.22) in M&S]

$$J^{\mu}(x) = J_1^{\mu}(x) - \mathrm{i}\, J_2^{\mu}(x), \qquad J^{\mu\dagger}(x) = 2J_1^{\mu}(x) + \mathrm{i}\, J_2^{\mu}(x), \tag{28}$$

where

$$J_1^{\mu}(x) = \frac{1}{2}\left[\bar{\psi}_{\nu_\ell}^{\mathrm{L}}\,\gamma^{\mu}\,\psi_{\ell}^{\mathrm{L}} + \bar{\psi}_{\ell}^{\mathrm{L}}\,\gamma^{\mu}\,\psi_{\nu_\ell}^{\mathrm{L}}\right], \quad J_2^{\mu}(x) = \frac{\mathrm{i}}{2}\left[\bar{\psi}_{\ell}^{\mathrm{L}}\,\gamma^{\mu}\,\psi_{\nu_\ell}^{\mathrm{L}} - \bar{\psi}_{\nu_\ell}^{\mathrm{L}}\,\gamma^{\mu}\,\psi_{\ell}^{\mathrm{L}}\right]. \tag{29}$$

ν_ℓ refers to the ℓ-leptonic neutrino. Only the left-handed leptons and neutrinos are relevant, by assumption [Section 17.2 in M&S]. $J^{\mu}(x)$ and $J^{\mu\dagger}(x)$ define the ladder operators I_-^{W} and I_+^{W} for the isospin, analogous to the ladder operatoes $S_\pm = S_x \pm \mathrm{i}\, S_y$ for the spin. They increase and decrease the values of the isospin charges by 1.

Now we want to determine the isospin charges for the leptonic states, analogously to what we did with the electromagnetic current. To do that, we choose the basis where I_3^{W} and $\left(I^{\mathrm{W}}\right)^2$ are simultaneously diagonalizable, as one does in quantum mechanics for the spin. However, we are assuming that the leptonic pairs (ℓ, ν_ℓ) (lepton and corresponding leptonic neutrino) are doublets under SU(2). Hence, we do not care about the eigenvalues of $\left(I^{\mathrm{W}}\right)^2$ because they are never changed in our context. Then, we only consider the third isospin current J_3^{μ} (which leads to the charge I_3^{W}, analogous to the spin operator S_z). The isospin neutral current is

$$J_3^{\mu} = \frac{1}{2}\left[\bar{\psi}_{\nu_\ell}^{\mathrm{L}}\,\gamma^{\mu}\,\psi_{\nu_\ell}^{\mathrm{L}} - \bar{\psi}_{\ell}^{\mathrm{L}}\,\gamma^{\mu}\,\psi_{\ell}^{\mathrm{L}}\right], \tag{30}$$

The isospin conserved neutral charge is (please check)

$$\begin{aligned} I_3^{\mathrm{W}} = \frac{1}{2}\sum_{\ell}\int d^3p\, \Big\{ &\Big[c_{2,\nu_\ell}^{\dagger}(\vec{p})\, c_{2,\nu_\ell}(\vec{p}) - d_{2,\nu_\ell}^{\dagger}(\vec{p})\, d_{2,\nu_\ell}(\vec{p})\Big] \\ &- \Big[c_{2,\ell}^{\dagger}(\vec{p})\, c_{2,\ell}(\vec{p}) - d_{2,\ell}^{\dagger}(\vec{p})\, d_{2,\ell}(\vec{p})\Big]\Big\}. \end{aligned} \tag{31}$$

We can then assign charges in the exact same way as in QED: the relative signs in I_3^{W} will determine them.

$$\begin{aligned} I_3^{\mathrm{W}}\,|\ell^-, L\rangle &= -\frac{1}{2}\,|\ell^-, L\rangle, \qquad & I_3^{\mathrm{W}}\,|\nu_\ell, L\rangle &= +\frac{1}{2}\,|\nu_\ell, L\rangle\,, \\ I_3^{\mathrm{W}}\,|\ell^+, L\rangle &= +\frac{1}{2}\,|\ell^+, L\rangle, \qquad & I_3^{\mathrm{W}}\,|\bar{\nu}_\ell, L\rangle &= -\frac{1}{2}\,|\bar{\nu}_\ell, L\rangle\,. \end{aligned}$$

All the right-handed particles have 0 isospin charge, by assumption (motivated by the experience).

Note. In quantum mechanics, one considers one particle only, and the spin operators change the spin of that particle. The isospin operators I_-^{W} and I_+^{W} raise and lower the isospin, and by doing so they change the particle itself. You can think of this as the left-handed leptons being different quantum states of a more general left-handed particle described by the doublet field $\Psi_\ell^{\mathrm{L}}(x) = \begin{pmatrix} \psi_{\nu_\ell}^{\mathrm{L}}(x) \\ \psi_\ell^{\mathrm{L}}(x) \end{pmatrix}$.

References

[Wei72] S. Weinberg. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*. July 1972, p. 688.