Additional notes on spontaneous symmetry breaking

FK8027 - Quantum Field Theory

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The spontaneous breaking of a symmetry of the Lagrangian is a deep and subtle concept. In QFT, we start with a Lagrangian defining our theory. If we apply a global transformation to the Lagrangian, and it stays the same in form and value, then we talk about a global symmetry of the Lagrangian. If this symmetry is local, then we talk about a gauge symmetry. In the following, we will consider global symmetries.

Now, we know that Noether's theorem (one of the most important theorem in Physics) guarantees "that the invariance under a continuous symmetry [...] implies the existence of a conserved current"[Str07, p. 162]. *However, the integrability of such a conserved current to a global unitary charge operator is not implied.* Actually, if such a current can be integrated to give rise to a charge operator, then the symmetry is unbroken or exact; if it can not, then the symmetry is spontaneously broken.

Let's deepen this concept by proceeding step by step. First, we introduce the Fabri–Picasso theorem, referring to Aitchison [Ait07, Sec. 6.1].

Theorem 1. (Fabri–Picasso) *Suppose that a given Lagrangian $\mathscr{L}$ is invariant under some one-parameter continuous global internal symmetry with a conserved Noether current j^μ, $\partial_\mu j^\mu = 0$. The associated charge is the Hermitian operator $Q = \int \mathrm{d}^3x\, j^0(x)$. Then, Q can act on the vacuum state of the theory $|0\rangle$ in* only *two ways:*

1. $Q\,|0\rangle = 0$,
2. $Q\,|0\rangle$ *does not exist in the same Hilbert space in which $|0\rangle$ is defined, because its norm is infinite.*

Proof. To prove the theorem, let's consider the norm of the state $Q\,|0\rangle$,

$$\begin{aligned}\langle 0|\,QQ\,|0\rangle &= \int \mathrm{d}^3x \int \mathrm{d}^3y\, \langle 0|\, j^0(x)\, j^0(y)\,|0\rangle \\ &= \int \mathrm{d}^3x \int \mathrm{d}^3y\, \langle 0|\, \mathrm{e}^{\mathrm{i}px} j^0(0) \mathrm{e}^{-\mathrm{i}px}\, \mathrm{e}^{\mathrm{i}py} j^0(0) \mathrm{e}^{-\mathrm{i}py}\,|0\rangle \\ &= \int \mathrm{d}^3x \int \mathrm{d}^3y\, \langle 0|\, j^0(0) \mathrm{e}^{-\mathrm{i}p(x-y)} j^0(0)\,|0\rangle\,. \end{aligned} \tag{1}$$

The first equality follows from how the charges transform under a translation in space–time; the second equality follows from $\mathrm{e}^{\mathrm{i}px}\,|0\rangle = |0\rangle$, since the vacuum state has $p = 0$.[1] Now we make a change of variable in the second integral, namely

$$z = y - x,\, \mathrm{d}^3 z = \mathrm{d}^3 y \Longrightarrow \langle 0|\, QQ\, |0\rangle = \int \mathrm{d}^3 x \int \mathrm{d}^3 z\, \langle 0|\, j^0(0) \mathrm{e}^{\mathrm{i}p(z)} j^0(0)\, |0\rangle\, . \tag{2}$$

We can perform the integral in $d^3 z$, obtaining a constant quantity $\mathcal{C}$, since the vacuum expectation value only depends on z, over which we integrate. We are left with

$$\langle 0|\, QQ\, |0\rangle = \int \mathrm{d}^3 x\, \mathcal{C}, \tag{3}$$

which is a divergent integral. Therefore, unless $\mathcal{C} = 0$, which is equivalent to $Q\,|0\rangle = 0$, the state $Q\,|0\rangle$ is not normalizable and, as such, does not exist. □

Note that, if $Q\,|0\rangle$ does not exist, the operator Q itself is ill-defined. In fact, in QFT we write any particle state as an operator acting on the vacuum state [Nai06, Sec. 11.1]

$$|\alpha\rangle = A_\alpha\, |0\rangle\, . \tag{4}$$

Now, suppose that $Q\,|0\rangle$ does not exist, but Q does exist. Then, the commutator $[Q, A_\alpha]$ exists, and we can act with Q on the state $|\alpha\rangle$,

$$Q\,|\alpha\rangle = Q\, A_\alpha\, |0\rangle = [Q, A_\alpha]\, |0\rangle + A_\alpha\, Q\, |0\rangle\, . \tag{5}$$

Since $Q\,|0\rangle$ is not defined, $Q\,|\alpha\rangle$ is not defined either $\forall\, |\alpha\rangle$. This means that the operator Q itself is ill-defined, i.e., it does not exist.

The situation is radically different if an operator annihilates the vacuum, as shown by the Coleman theorem, which we only state here without proof [Ait07, Sec. 6.1].

Theorem 2. (Coleman) *Suppose to have an operator* $Q(t) = \int \mathrm{d}^3 x\, j^0(x)$*,* without *the requirement* $\partial_\mu j^\mu = 0$*. If* $Q(t)\,|0\rangle = 0$*, then*

1. $\partial_\mu j^\mu = 0$*, so the vector* j^μ *is actually conserved,*
2. $Q(t) = Q$ *is independent of time,*
3. *there exist a unitary operator* $U = \mathrm{e}^{\mathrm{i}\lambda Q}$ *acting on the Hilbert space of quantum states, which defines the action of the symmetry on the quantum states.*

[1] The vacuum state is the *unique* state which is invariant under a Poincaré transformation.

Summarizing, if $Q(t)|0\rangle = 0$, then an associate conserved current exists and $Q(t)$ is time-independent, therefore it is associated with a Noether symmetry of the Lagrangian. Moreover, Q defines a unitary operator $U = \mathrm{e}^{\mathrm{i}\lambda Q}$ acting on the quantum states, and therefore the symmetry having Q as the charge operator is exact. On the other hand, if $Q|0\rangle$ does not exist, then the charge operator Q itself does not exist, and the symmetry cannot be implemented unitarily in the Hilbert space of quantum states, thus being spontaneously broken.

Quoting again, "A global charge as algebraic generator [...] does not exist if the symmetry is broken (we have already remarked that the formal integral of j_0 does not define an operator), and one can only speak of a local generation [...] in terms of local charges" [Str07, p. 162].

Now, suppose we have a spontaneously broken symmetry. Then, we saw that we cannot define a unitary operator implementing the symmetry on the Hilbert space of quantum states. However, this is still a symmetry of the Lagrangian,[2] which means that we must be able to compute the variation of the fields in the Lagrangian under the symmetry. Otherwise, we could not even check that the Lagrangian is invariant. However, if the charge operator does not exist, how can we compute the variation of the fields? The answer is the following.

The Noether theorem says that $\forall A$ local operator, the variation δA of the operator under a symmetry of the Lagrangian (exact or spontaneously broken) is given by [Str07, p. 167]

$$\delta A(\vec{y}, t_0) = \mathrm{i} \lim_{R\to\infty} [Q_R(t_0), A(\vec{y}, t_0)] = \mathrm{i} \lim_{R\to\infty} \int_{|\vec{x}|<R} \mathrm{d}^3x \left[j^0(\vec{x}, t_0), A(\vec{y}, t_0)\right], \tag{6}$$

with $\partial_\mu j^\mu = 0$. The integral of the commutator $\left[j^0(\vec{x}, t_0), A(\vec{y}, t_0)\right]$ has much better convergence property than the integral of $j^0(\vec{x}, t)$ only, since the commutator has a compact support if both $j^0(\vec{x}, t_0)$ and $A(\vec{y}, t_0)$ are local—this is an equal-time commutator which is non-zero only for timelike and null separations between x and y, therefore the integration has support on the compact region $x^2 + y^2 + z^2 \leq t_0^2$, with t_0 constant [Str07, p. 166]. Consequently, the variation δA of a field can be computed even if the symmetry is spontaneously broken and the charge operator is not well-defined.

We now introduce a more precise definition for a spontaneously broken symmetry, in the light of the previously discussion. Since, as we saw, we can always compute the variation of a local field operator A under a symmetry of the Lagrangian, we denote the transformed field with A'. The definition of a spontaneously broken symmetry is then [Str07, pp. 120,121]

$$\exists A : \langle 0| A' |0\rangle \neq \langle 0| A |0\rangle . \tag{7}$$

[2]It is called an *algebraic symmetry*, see [Str07, p. 119]

Conversely, a symmetry is exact iff

$$\forall A : \langle 0| A' |0\rangle = \langle 0| A |0\rangle \,. \tag{8}$$

This means that an exact symmetry does not change the expectation values, whereas a spontaneously broken symmetry does change them. Therefore, a spontaneously broken symmetry is not a symmetry for the observable quantities, hence it is not realized in the "real world". Now, we can rewrite (7) as,

$$\exists A : 0 \neq \langle 0| \left\{ A' - A \right\} |0\rangle = \langle 0| \delta A |0\rangle \,. \tag{9}$$

We know from (6) that

$$\langle 0| \delta A(\vec{y}, t_0) |0\rangle = \mathrm{i} \lim_{R\to\infty} \int_{|\vec{x}|<R} \mathrm{d}^3 x \, \langle 0| \left[j^0(\vec{x}, t_0), A(\vec{y}, t_0) \right] |0\rangle \,. \tag{10}$$

We saw that, if the symmetry is exact, then the charge exists globally and annihilates the vacuum. Therefore $\langle 0| \delta A |0\rangle$ converges to $\mathrm{i} \langle 0| [Q, A] |0\rangle$ and it is trivially zero for any A, since $Q |0\rangle = 0$. On the other hand, if the symmetry is broken, when $R \to \infty$ this integral converges to a finite value and not to zero.

This implies that $\langle 0| \delta A |0\rangle \neq 0$, i.e., there is a field operator with a non-zero vacuum expectation value (in our case, the Higgs field). The vacuum expectation value $\langle 0| A |0\rangle$ is called "symmetry breaking order parameter" [Str07, p. 121].

References

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