Tutorial 4

FK8027 - Quantum Field Theory

Monday 25th November, 2019

Topics for today

- Invariance in form and value
- Rotations in 3D euclidean space and the definition of a group
- The Lorentz group
- The unitary group
- The generators of a group of differentiable transformations
- The importance of symmetries in physics

1 Invariance in form and value

An important concept which is related to that of "symmetries of a system" is the concept of invariance. If we have a system which is invariant under a certain transformation, then we say that the transformation is a symmetry for that system. Being invariant means that the system stays the same, is equal to itself before and after the transformation. At this point, it is important to emphasize the difference between two types of invariance that we can encounter: invariance "in form" and invariance "in value".

Consider the scalar product of two vectors. It is invariant in value under generic coordinate transformations (GCT). We know that from basic geometry. This means the following: Take two vectors v^μ and u^μ written in certain coordinates, and compute the scalar product between them. You get the value $g_{\mu\nu}u^\mu v^\nu = a$. Now do a GCT; the vectors will transform (contravariantly) and their components will be different. However, if you compute their scalar product in the new coordinate system, you get $g'_{\mu\nu}u'^\mu v'^\nu = a$ again. This is invariance in value, because the scalar product is equal to a before and after the transformation, and invariance in form, since the expression of the scalar product stays the same, but involves the components of the objects in the appropriate coordinate systems. Invariance in form is also called "covariance".

Now consider the special relativistic Newton's second law of mechanics in the Cartesian coordinate system, for a particle with mass $m = 1$ [1, Ch. 11],

$$\frac{dx^\mu}{d\tau^2} = F^\mu, \tag{1}$$

with τ proper time of the particle and F^μ four-force. Let's apply a GCT from x^μ to $\tilde{x}^\mu(x)$ and compute the four-force and the four-acceleration in the new coordinate system,

$$\begin{aligned}
\tilde{F}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} F^\nu = \frac{d^2 \tilde{x}^\nu}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{d\tilde{x}^\mu(x)}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right) \\
&= \frac{dx^\nu}{d\tau}\frac{d}{d\tau}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right) + \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2} = \frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\rho}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right)\frac{dx^\rho}{d\tau} + \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2} \\
&= \frac{dx^\nu}{d\tau}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\nu \partial x^\rho}\frac{dx^\rho}{d\tau} + \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2}.
\end{aligned} \tag{2}$$

Now contract the free index μ with $\partial x^\sigma / \partial \tilde{x}^\mu$,

$$\begin{aligned}
\frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial \tilde{x}^\mu}{\partial x^\nu} F^\nu &= \frac{dx^\sigma}{d\tilde{x}^\mu}\frac{dx^\nu}{d\tau}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\nu \partial x^\rho}\frac{dx^\rho}{d\tau} + \frac{dx^\sigma}{d\tilde{x}^\mu}\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\frac{d^2 x^\nu}{d\tau^2} \\
\delta^\sigma{}_\nu F^\nu &= \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{dx^\nu}{d\tau}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\nu \partial x^\rho}\frac{dx^\rho}{d\tau} + \delta^\sigma{}_\nu \frac{d^2 x^\nu}{d\tau^2} \\
F^\sigma &= \boxed{\left(\frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\nu \partial x^\rho}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}} + \frac{d^2 x^\sigma}{d\tau^2}.
\end{aligned} \tag{3}$$

We see in (3) that the four-acceleration is not anymore equal to the four-force, as it was in (1) in the Cartesian coordinate system, due to the boxed term. This physical law is not invariant in form under a GCT, nor in value since, for example, $\tilde{F}^\mu \neq F^\mu$. If the laws of physics are not invariant in form under a GCT, then this means that two observers experience different physical phenomena. This is not a desirable property for a physical theory. Such things already appear in classical mechanics when studying apparent forces; here, we are dealing with the same phenomenon. A GCT does not necessarily send inertial frames to inertial frames, hence apparent forces manifest and are described by the boxed term in (3). If we restrict ourselves to GCT such that $\partial^2 \tilde{x}^\mu / \partial x^\nu \partial x^\rho = 0$, then the expression is invariant in form.[1]

A theory is said to be "covariant" under a transformation (not necessarily a GCT) if its fundamental mathematical objects (action, equations etc.) are invariant in form under that transformation. A theory is said to be "invariant" under a transformation, if if its fundamental mathematical objects are invariant in form and value under that transformation. In the latter case, the transformation is said to be a "symmetry" of the theory.

In the following, we will be interested in transformations that keep physical quantities invariant in form and value.

[1] Note that we just showed that special relativity, hence QFT, are not invariant in form under a GCT. They describe physics only for inertial observers and do not include apparent forces. In order to describe physics also for accelerated observers in a systematic manner, general relativity is needed.

2 The orthogonal group of euclidean rotations

We have seen that, in the relativistic notation (contravariant and covariant indices), the scalar product between two vectors v^μ and u^μ can be written as

$$\vec{u} \cdot \vec{v} = u_\mu v^\mu = g_{\mu\nu} u^\mu v^\mu. \tag{4}$$

Then the squared norm of a vector can be defined as

$$\vec{v} \cdot \vec{v} = g_{\mu\nu} v^\mu v^\mu. \tag{5}$$

In the familiar 3-dimensional euclidean space, we have

$$g_{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

that is, the metric is represented by the identity matrix. The scalar product is then,

$$v_\mu u^\mu = v_1 u^1 + v_2 u^2 + v_3 u^3 = v^1 u^1 + v^2 u^2 + v^3 u^3. \tag{7}$$

We know that the squared norm of a vector does not change in form and value under a spatial rotation described by the matrix $R^\mu{}_\nu$.[2] We can write this in the relativistic notation,

$$v'^\mu = R^\mu{}_\nu v^\nu, \qquad \delta_{\mu\nu} v^\mu v^\nu = \delta_{\mu\nu} v'^\mu v'^\nu = \delta_{\mu\nu} R^\mu{}_\rho v^\rho R^\nu{}_\sigma v^\sigma, \tag{8}$$

which can be rewritten as

$$\delta_{\rho\sigma} v^\rho v^\sigma = \left[\left(R^\intercal\right)_\rho{}^\mu \delta_{\mu\nu} R^\nu{}_\sigma \right] v^\rho v^\sigma. \tag{9}$$

This must hold for every vector v^μ, therefore we have

$$\delta_{\rho\sigma} = \left(R^\intercal\right)_\rho{}^\mu \delta_{\mu\nu} R^\nu{}_\sigma. \tag{10}$$

In matrix notation

$$\delta = R^\intercal \, \delta \, R. \tag{11}$$

Note that this implies $R^\intercal = R^{-1}$. The usual way to introduce spatial rotations is to *define* the matrix R through (11). A matrix fulfilling (11) is said to be "orthogonal", and the set of all possible orthogonal matrices of dimension $n \times n$ is called $O(n)$. In our case, the dimension of the space is

[2] The scalar product depends on the moduli of the two vectors and the relative angle between them, which does not change if we rotate both vectors by the same angle.

three and then the set is called $O(3)$. If we take the determinant of both sides of (11), we get

$$\det(\delta) = \det(R^{\intercal}\,\delta\,R) = \det(R^{\intercal})\det(\delta)\det(R)\,. \tag{12}$$

Since $\det(R^{\intercal}) = \det(R)$, this implies,

$$\det(R) = \pm 1. \tag{13}$$

If the determinant is $+1$, we are considering proper rotations (i.e., continuous rotations); if the determinant is -1, we are considering reflections, which are not continuous, but discrete transformations. The set of all matrices satisfying (11) and having determinant $+1$ is called $SO(n)$ in n-dimensions, that is "special orthogonal".

The set $O(n)$ is a "group", a concept which we are going to define now.

Definition. A group is a non-empty set G, together with an operation $\circ$ (called the "group law") which takes two generic elements g_1, g_2 of G and associates another element $g_1 \circ g_2$ to them. The group law must be such that the following properties hold:

- **Closure.** $g_1 \circ g_2 \in G, \quad \forall\, g_1, g_2 \in G.$
- **Associativity.** $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3, \quad \forall\, g_1, g_2 \in G.$
- **Identity element.** There exist an element e such that

  $$g \circ e = e \circ g = g, \quad \forall\, g \in G.$$

  e is called the "identity element" of the group, and it can be shown that, if it exists, it is *unique*.
- **Inverse element.** For each $g \in G$, there exist an element $h \in G$, commonly denoted with g^{-1} or $-g$, such that $g \circ h = h \circ g = e$.

These four properties are called the "group axioms". Note that we are *not* guaranteed that the group law is commutative generically, i.e.

$$g_1 \circ g_2 \neq g_2 \circ g_1. \tag{14}$$

If the group law is commutative, then the group is called "abelian"; otherwise, it is called "non-abelian". A simple example of an abelian group is the real line $\mathbb{R}$ with the addition $+$ as the group law (please check).

As we can easily accept, the set of orthogonal matrices $O(3)$ with the matrix product as the group law, is a group. If we apply two consecutive improper rotations (meaning reflections) or a proper rotation followed by an improper one, R and T, to the space, the result of this operation is another improper rotation S. In formulas,

$$T \cdot R = S, \tag{15}$$

where $\cdot$ denotes the matrix product. If we apply two consecutive proper rotations, the result is another proper rotation. This means that $O(3)$ is closed. It is associative because the matrix product is such. The identity element is simply the rotation by a zero angle. The inverse element of a rotation by θ is simply the rotation by $-\theta$, and the inverse element of a reflection is the opposite reflection. Therefore, from now on, we will talk about the $O(3)$ orthogonal *group*. $SO(3)$ is also a group, and in particular it is a "subgroup" of $O(3)$.

Definition. A subgroup H of a group G is a subset of G which is a group under the same group law of G.

Therefore, we will also talk about the special orthogonal group, which is the group of proper (i.e., continuous) rotations. The orthogonal group $O(3)$ and the special orthogonal group $SO(3)$ are very important in classical mechanics. All classical mechanical laws have to be invariant under rotations, and this can be mathematically stated by saying that the physical laws have to be invariant under a transformation belonging to $O(3)$.[3] We then say that spatial rotations are symmetries for classical mechanics.

3 The Lorentz group

In QFT, we use special relativity and therefore the metric is not the identity. It is the Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{16}$$

as we already know. Therefore, the scalar product is different,

$$\begin{aligned} \vec{v} \cdot \vec{u} = v_\mu u^\mu = \eta_{\mu\nu} v^\mu u^\mu &= v^0 u^0 - v^1 u^1 - v^2 u^2 - v^3 u^3 \\ &= v_0 u^0 + v_1 u^1 + v_2 u^2 + v_3 u^3. \end{aligned} \tag{17}$$

We are interested in finding the symmetries of special relativity. We then consider a transformation $L^\mu{}_\nu$ acting on a four–vector v^μ, and we *require* that the Minkowski scalar product is kept invariant in form and value,

$$v'^\mu = L^\mu{}_\nu v^\nu, \qquad \eta_{\mu\nu} v^\mu v^\nu = \eta_{\mu\nu} v'^\mu v'^\nu = \eta_{\mu\nu} L^\mu{}_\rho v^\rho L^\nu{}_\sigma v^\sigma, \tag{18}$$

Similarly to the euclidean case, we can write

$$\eta_{\rho\sigma} = (L^\intercal)_\rho{}^\mu \eta_{\mu\nu} L^\nu{}_\sigma, \tag{19}$$

[3] For the more interested people, the laws of classical mechanics have to be invariant also under translations and uniform motions, i.e. they have to be same in any inertial frame; look for "Galilean group".

or, in matrix notation,

$$\eta = L^{\intercal}\,\eta\,L. \tag{20}$$

The equation (20) is the *definition* of "Lorentz transformations". They are transformations preserving the Minkowski scalar product or, equivalently, the Minkowski metric. If we multiply both sides by -1, then $\eta_{\mu\nu} \to -\eta_{\mu\nu}$ and $L^{\mu}{}_{\nu}$ is unchanged. This shows that the signature is not important for our purposes. An analogous calculation as the one done for $O(3)$ shows us that

$$\det(L) = \pm 1. \tag{21}$$

These transformations also form a group, called the "Lorentz group" $O(1,3)$, that is, the "pseudo–orthogonal" group for a metric with signature $(+,-,-,-)$ [or $(-,+,+,+)$]. We studied some properties of this group also in previous tutorials (see Handouts for Tutorial 1 and 2). The Lorentz group is then the symmetry group of special relativity (and QFT).

4 The unitary group

So far, we have introduced two real groups. Now we talk about a complex group. We define a "unitary" matrix U as a matrix satisfying

$$U^{\dagger}\,\delta\,U = U\,\delta\,U^{\dagger} = \delta, \tag{22}$$

where $\dagger$ means "transpose conjugate". Then we immediately see that, when we apply U to a complex vector z (i.e., a vector with complex components; in quantum mechanics this vector is a ket, and its adjoint is a bra), we get (in matrix notation)

$$z' = Uz, \qquad z'^{\dagger}z' = (Uz)^{\dagger}\,Uz = z^{\dagger}U^{\dagger}Uz = z^{\dagger}z, \tag{23}$$

so the unitary matrices preserve in form and value the inner product of quantum mechanics between bras and kets (just replace z by $|\lambda\rangle$ and $z^{\dagger}$ with $\langle\lambda|$). The set of unitary matrices form another group, called the unitary group $U(n)$ in n *complex* dimensions. From (22) we obtain,

$$U^{\dagger} = U^{-1}, \qquad |\det(U)| = 1 \Rightarrow \det(U) = \mathrm{e}^{\mathrm{i}\phi}. \tag{24}$$

Again, if we impose $\det(U) = 1$, then we have a subgroup of the unitary group, called the "special unitary" group $SU(n)$.

We now compute the number of independent *real* components of a unitary matrix. A $n \times n$ complex matrix has n^2 complex components, that is, $2n^2$ real components (real and imaginary part). If we impose that it is unitary, we have to impose the equations in (22). A generic $n \times n$ complex matrix can be written as,

$$U = \begin{pmatrix} a & \dots & b \\ \vdots & \ddots & \vdots \\ c & \dots & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{C}. \tag{25}$$

Then (22) reads

$$UU^\dagger = \begin{pmatrix} a & \dots & b \\ \vdots & \ddots & \vdots \\ c & \dots & d \end{pmatrix} \cdot \begin{pmatrix} a^* & \dots & c^* \\ \vdots & \ddots & \vdots \\ b^* & \dots & d^* \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \tag{26}$$

which is equal to

$$\begin{pmatrix} |a|^2 + \dots + |b|^2 & \dots & a\,c^* + \dots + b\,d^* \\ \vdots & \ddots & \vdots \\ c\,a^* + \dots + d\,b^* & \dots & |c|^2 + \dots + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}. \tag{27}$$

The diagonal equations are real, so they count as n equations. The non-diagonal equations are complex, so they count as two equations each (the real and the imaginary part have to be zero separately). However, we can see that the equation (ji) is the conjugate of the equation (ij), therefore we have to consider only the upper triangle. Hence we have d independent equations, where d is

$$d = \#^{\text{eqs}}_{\text{real,diagonal}} + \#^{\text{eqs}}_{\text{real,nondiagonal}} = n + \left(\frac{n^2 - n}{2}\right) 2 = n^2. \tag{28}$$

Therefore, the number of independent components of a unitary matrix is

$$2n^2 - d = 2n^2 - n^2 = n^2. \tag{29}$$

For a matrix in $SU(n)$, the independent components are $n^2 - 1$, because we also impose one more equation, namely that the determinant is 1.

For $n = 2$, a matrix in $U(2)$ has 4 independent components, or “degrees of freedom”, whereas a matrix in $SU(2)$ has 3 degrees of freedom.

5 The generators of a group of differentiable transformations

The concepts of generators of a group has a deep meaning and can be associated with every group whose elements are differentiable transformations [for example, to $SO(n), SO(n,m), SU(n)$]. We will talk about the generators of $SU(2)$, but it is important to know that an analogous analysis can be carried out for other groups too.

We saw that a matrix in $SU(2)$ has three degrees of freedom. Naïvely, one would then think that it can be expressed in a basis of matrices having three basis elements. However, $SU(2)$ is a group and not a vector space, so we have to work a bit more. For matrix groups of differentiable transformations, we can write a transformation g as

$$g = \mathrm{e}^{\mathrm{i}\,T}, \tag{30}$$

For $SU(2)$, we can write

$$U = \mathrm{e}^{\mathrm{i}\,H}. \tag{31}$$

We know that this U has to satisfy (22), so

$$UU^\dagger = \mathrm{e}^{\mathrm{i}\,H}\,\mathrm{e}^{-\mathrm{i}\,H^\dagger} = \mathrm{e}^{\mathrm{i}\left(H-H^\dagger\right)} = \mathbb{1}, \tag{32}$$

with $\mathbb{1}$ the identity matrix, which implies

$$H = H^\dagger, \tag{33}$$

that is, H is hermitian. So a unitary matrix can be expressed as the exponential of a hermitian matrix. A special unitary matrix must have determinant 1, which means

$$1 = \det(U) = \det\left(\mathrm{e}^{\mathrm{i}\,H}\right) = \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}(H)} \Longrightarrow \mathrm{Tr}(H) = 0, \tag{34}$$

where we used the Jacobi's formula for the determinant of the exponential of a matrix.

It turns out that the set of hermitian matrices with the matrix sum is a vector space over the complex field (the same if they are also traceless). Therefore, we can express any hermitian matrix as

$$H = \theta_1 H_1 + \theta_2 H_2 + \theta_3 H_3, \tag{35}$$

where the H_i constitute a basis in the vector space and the θ_i are *complex parameters which can depend on the coordinates* x^μ *or not.* Therefore, a generic element of $SU(2)$ is

$$U = \mathrm{e}^{\mathrm{i}(\theta_1 H_1 + \theta_2 H_2 + \theta_3 H_3)} \tag{36}$$

Varying the θ_i means varying H, hence varying the group element U, that is, the transformation. The three matrices H_i are called the "generators" of $SU(2)$ (in general, the generators of the group one is considering). They are three for $SU(2)$ because it has three degrees of freedom.

The generators of $SU(2)$ are then three complex 2-dimensional hermitian traceless linearly independent matrices. We can recognize the Pauli matrices as having these properties,

$$\sigma_1 \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{37}$$

For the continuous transformations of $U(2)$, everything is the same, but we do not impose anymore the determinant to be 1, and so we have four independent components and we need one more matrix in our basis. The new matrix is simply the 2-dimensional identity, which is linearly independent from the Pauli matrices and it is not traceless.

6 The importance of symmetries in physics

Why is all of this important? And is there any difference between the Lorentz group, which is tight to the metric of spacetime, and the unitary group?

First, we answer to the second question. Let's consider the Lorentz group. We saw that special relativity is not covariant under a GCT. That means that a GCT is not a symmetry of QFT, but a Lorentz transformation is, since it is linear, implying $\partial^2\tilde{x}^\mu/\partial x^\nu \partial x^\rho = 0$ in (3). In other words, the action of the theory is invariant under a Lorentz transformation ("Lorentz-invariant"), and consequently the equations of motion are covariant under a Lorentz transformation ("Lorentz-covariant"). In the case of a GCT in QFT, instead, the action of the theory is not invariant (it is still the same in value, because it is a scalar, but it is not covariant), meaning that the equations of motion also change. Note that the Lorentz transformations are actually symmetries of the spacetime, meaning that they act on the spacetime itself. The rotations and boosts move the coordinate frames of the spacetime.

Let's turn to the unitary group. Usually, unitary transformations are symmetries of the action, but not of the spacetime. Indeed, the action can have more symmetries than the spacetime. Consider the following action for a massless complex scalar field ϕ,

$$\int \mathrm{d}^4x\, \partial_\mu \phi^\dagger(x) \partial^\mu \phi(x). \tag{38}$$

Suppose we apply a $U(1)$ transformation to the *field* itself. A $U(1)$ transformation is described by a simple phase factor $e^{\mathrm{i}\theta}$, with θ constant. The action becomes,

$$\int \mathrm{d}^4x\, e^{-\mathrm{i}\theta} e^{\mathrm{i}\theta}\, \partial^\mu \phi^\dagger(x) \partial_\mu, \phi(x), \tag{39}$$

since $\phi(x) \to e^{\mathrm{i}\theta}\phi(x)$, $\phi^\dagger(x) \to e^{-\mathrm{i}\theta}\phi(x)$. Note that, if θ was not constant, this would have required a bit more work. So we get the result that the action stays the same under this transformation of the field, not of the spacetime. This kind of symmetries is called "internal symmetries" of the theory, or of the action.

If you have more than one field which are connected by an internal symmetry of the action, then they are referred to as a "multiplet". For example, suppose there is a theory describing the dynamics of three massive complex fields ϕ_1, ϕ_2, ϕ_3 with the same mass, with action

$$\int \mathrm{d}^4x \left(\partial_\mu \Phi^\dagger(x) \partial^\mu, \Phi(x) - m^2\, \Phi^\dagger \Phi \right), \tag{40}$$

where $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$ and $\Phi^\dagger = (\phi_1^* \quad \phi_2^* \quad \phi_3^*)$.[4] This action is invariant under a $U(3)$ transformation U which acts in the following way (please verify),

$$\Phi' = \begin{pmatrix} \phi_1' \\ \phi_2' \\ \phi_3' \end{pmatrix} = U \cdot \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = U \cdot \Phi, \tag{41}$$

where the ϕ_i' are linear combinations of the ϕ_i. Then we say that the three fields constitute a multiplet under the group $U(3)$, or a $U(3)$-multiplet. Note that the fields in a multiplet with respect to any group of symmetries need to have the same mass, otherwise the action would not be invariant under the transformation.

Now we answer to the first question. All of this is extremely important in physics because of Noether's (first) theorem, which states that, for any continuous symmetry of the action of a physical system, there is an associated conservation law. Physically, this means that there can be a conserved charge associated with a given symmetry of the action.[5] We will see that the Standard Model of particle physics is invariant under a certain group of internal symmetries (in addition to being invariant under the Lorentz symmetries), and this implies the existence of a certain number of conserved charges (the electric charge, for example).[6]

References

[1] M. Tsamparlis, *Special relativity: an introduction with 200 problems and solutions* (Springer Berlin Heidelberg, 2010) (cit. on p. 1).

[4] If you expand this action in terms of ϕ_1, ϕ_2, ϕ_3 and the complex conjugates, you will get three copies of (38), but with the mass terms.

[5] There can be cases where a conservation law does not imply the presence of a conserved charge, but we will treat this case very much later in this course.

[6] The invariant charges associated with the Lorentz group are the four-momentum and the total angular momentum (orbital plus spin).