Tutorial 6

FK8027 - Quantum Field Theory

Monday 9$^{\text{th}}$ December, 2019

Topics for today

- Derivation of the Dirac equation from the Klein–Gordon one
- Derivation of the Dirac equation from the Dirac action
- Lorentz covariance of the Dirac equation
- The solution to the Dirac equation

Some of these topics are treated in [M&S, Appendix A]. The Dirac equation is

$$\left(\mathrm{i}\hbar\,\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi(x) = 0. \tag{1}$$

In natural units, $\hbar = c = 1$, and with the Feynman's slashed notation $\partial\!\!\!/ \coloneqq \gamma^\mu \dfrac{\partial}{\partial x^\mu}$, it can be rewritten as

$$\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi(x) = 0. \tag{2}$$

The Dirac lagrangian is,

$$L = c\overline{\psi}\left(\mathrm{i}\hbar\,\gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right)\psi = \overline{\psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi, \tag{3}$$

with $\overline{\psi} \coloneqq \psi^\dagger\gamma^0$ being the "Dirac conjugate" of ψ.

1 Derivation of the Dirac equation from the Klein–Gordon one

Dirac's motivation to find his equation relied on the fact that the Klein–Gordon equation has both positive energy and negative energy solutions. At first, before the interpretation concerning positive and negative frequencies came out, this was seen as a problem. Dirac was trying to find an equation which can be viewed as the "square root" of the Klein–Gordon equation, in order to eliminate the potentially "unphysical" (at that time) solutions. However, he knew he could not obtain something which violates the Klein–Gordon equation, because the latter is an expression of the relativistic mass-shell condition $E^2 = |\vec{p}|^2 + m^2$.

Then, he did the following. Suppose that we are looking for an equation of the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left(c\vec{\alpha}\cdot\vec{p} + mc^2\beta\right)\psi, \tag{4}$$

with each of the α^i and β being constant matrices, $\vec{p} = -\mathrm{i}\hbar\nabla$, $\vec{\alpha} = \vec{\alpha}^\dagger$, $\beta = \beta^\dagger$, $[\vec{\alpha}, \vec{p}] = [\beta, \vec{p}] = 0$, the latter condition meaning that the α^i and β are independent from $\vec{p}$. Let's apply twice the differential operator $\mathrm{i}\hbar\dfrac{\partial}{\partial t}$ to ψ and impose that the result is the Klein–Gordon equation,

$$(\mathrm{i}\hbar)^2 \frac{\partial\psi}{\partial t^2} = \left(c\vec{\alpha}\cdot\vec{p} + mc^2\beta\right)^2 \psi, \tag{5a}$$

$$-\hbar^2 \frac{\partial\psi}{\partial t^2} = \left[c^2\left(\vec{\alpha}\cdot\vec{p}\right)^2 + m^2c^4\beta^2 + mc^3\vec{\alpha}\cdot\vec{p}\beta + mc^3\beta\vec{\alpha}\cdot\vec{p}\right]\psi \tag{5b}$$

$$= \left[-c^2\hbar^2\left(\alpha^i p_i\right)^2 + m^2c^4\beta^2 + mc^3\left(-\mathrm{i}\hbar\alpha^i\partial_i\right)\beta + mc^3\beta\left(-\mathrm{i}\hbar\alpha^i\partial_i\right)\right]\psi. \tag{5c}$$

We can rewrite this equation as

$$\left[-\hbar^2\frac{\partial^2\psi}{\partial t^2} + c^2\hbar^2\left(\alpha^i\partial_i\right)^2 - m^2c^4\beta^2\right]\psi = -\left[mc^3\hbar\partial_i\left\{\alpha^i, \beta\right\}\right]\psi, \tag{6a}$$

$$\left[\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \left(\alpha^i\partial_i\right)^2 + \frac{m^2c^2}{\hbar^2}\beta^2\right]\psi = \frac{mc}{\hbar}\left[\partial_i\left\{\alpha^i, \beta\right\}\right]\psi. \tag{6b}$$

In order for the last equation to be the Klein–Gordon equation, we must impose the following conditions

1. $\left(\alpha^i\partial_i\right)^2 = \alpha^i\partial_i\alpha^j\partial_j \overset{!}{=} \mathbb{1}\partial_k\partial^k = \nabla^2$
2. $\beta^2 \overset{!}{=} \mathbb{1}$
3. $\left\{\alpha^i, \beta\right\} \overset{!}{=} 0$

We impose the first condition,

$$\begin{aligned}\alpha^i\partial_i\alpha^j\partial_j = \alpha^i\alpha^j\partial_i\partial_j &= \frac{1}{2}\left(\left[\alpha^i, \alpha^j\right] + \left\{\alpha^i, \alpha^j\right\}\right)\partial_i\partial_j \\ &= \frac{1}{2}\left\{\alpha^i, \alpha^j\right\}\partial_i\partial_j \overset{!}{=} \mathbb{1}\partial_i\partial^i.\end{aligned} \tag{7}$$

This implies that

$$\frac{1}{2}\left\{\alpha^i, \alpha^j\right\} = \delta^{ij}\mathbb{1} \Longrightarrow \mathbb{1} = \frac{1}{2}\left\{\alpha^i, \alpha^i\right\}\mathbb{1} = \left(\alpha^i\right)^2 = \mathbb{1}. \tag{8}$$

So we get that $\left(\alpha^i\right)^2 = \beta^2 = \mathbb{1}$. Let's compute the trace of these matrices,

$$\mathrm{Tr}\left(\beta\right) = \mathrm{Tr}\left(\left(\alpha^i\right)^2\beta\right), \text{ anticommutation between } \alpha^i \text{ and } \beta \text{ gives,} \tag{9a}$$

$$= -\,\mathrm{Tr}\left(\alpha^i\beta\alpha^i\right), \text{ cyclic invariance gives,} \tag{9b}$$

$$= -\,\mathrm{Tr}\left(\alpha^i\alpha^i\beta\right) = -\,\mathrm{Tr}\left(\beta\right). \tag{9c}$$

It follows that $\mathrm{Tr}\left(\beta\right) = 0$. For the same reason, it also holds $\mathrm{Tr}\left(\alpha^i\right) = 0$ (please check it). So we need four traceless hermitian matrices with unitary real eigenvalues (since their square is the identity); this means that their dimension must be even, because the number of 1 and -1 on the diagonal must be the same in order the trace to be 0. In 2 dimensions the only matrices which satisfy these properties are the Pauli matrices σ^i, but they are only three and we need four of them. Then we must go to dimension 4; a possible set of matrices is

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{10}$$

You can verify that these matrices satisfy all the needed properties. At this point we define the Dirac γ matrices as

$$\gamma^0 \equiv \beta, \quad \gamma^i \equiv \beta\alpha^i, \quad \gamma^\mu \equiv \left(\gamma^0, \gamma^i\right). \tag{11}$$

The Dirac equation in natural units becomes

$$\mathrm{i}\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + m\beta)\,\psi. \tag{12}$$

We now multiply from the left by γ^0, which is invertible, so we do not lose information,

$$\mathrm{i}\gamma^0\frac{\partial\psi}{\partial t} = \left(\gamma^0\vec{\alpha}\cdot\vec{p} + m\gamma^0\gamma^0\right)\psi, \tag{13a}$$

$$\mathrm{i}\gamma^0\frac{\partial\psi}{\partial t} = \left(-\mathrm{i}\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi, \tag{13b}$$

$$\left(\mathrm{i}\gamma^0\frac{\partial}{\partial x^0} + \mathrm{i}\gamma^i\frac{\partial}{\partial x^i} - m\right)\psi = 0, \tag{13c}$$

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi = 0. \tag{13d}$$

As we know, the Dirac equation does have negative energy solutions, contrary to Dirac's hope. However, a posteriori, this is an excellent thing because it allows us to describe antifermions.

Exercise 1. Prove that if $\psi(x)$ satisfies the Dirac equation, it is also a solution to the Klein–Gordon equation.

Solution. If $\psi(x)$ satisfies the Dirac equation, then,

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi(x) = 0. \tag{14}$$

Now we apply the conjugate Dirac operator to this equation from the left,

$$\begin{aligned} 0 &= (-\mathrm{i}\gamma^\nu\partial_\nu - m)\,(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi(x) \\ &= \left(\gamma^\nu\gamma^\mu\partial_\mu\partial_\nu + m^2\right)\psi(x). \end{aligned} \tag{15}$$

Since $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, only the symmetric part of $\gamma^\mu \gamma^\nu$ is relevant, so

$$\begin{aligned} 0 &= \left(\gamma^{(\nu}\gamma^{\mu)}\partial_\mu\partial_\nu + m^2\right)\psi(x) \\ &= \left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\psi(x) = \left(\partial^\mu\partial_\mu + m^2\right)\psi(x) = \left(\Box + m^2\right)\psi(x), \end{aligned} \tag{16}$$

i.e., $\psi(x)$ also satisfies the Klein–Gordon equation. It has to be like this, since we obtained the Dirac equation by imposing the Klein–Gordon equation in this section. Actually, each of the four-component of the Dirac spinor $\psi(x)$ satisfies the Klein–Gordon equation separately.

What does this mean? Why do we need the Dirac equation, if all of its solutions satisfy the Klein–Gordon equation as well? Since the Dirac equation has negative energy solutions as well, couldn't we just use the Klein–Gordon equation?

Since we need a way to describe fermions, which cannot be mathematically represented by ordinary tensors, the introduction of spinors is necessary. The additional properties that fermions have, namely the fact that they do not rotate into themselves after a 2π rotation, has to be described by the spinors, which then need more structure than tensors to account for it. This additional structure is provided if a field solves the Dirac equation, not only the Klein–Gordon equation. Indeed, a solution to the Klein–Gordon equation does not necessarily satisfy the Dirac equation, meaning that it is not necessarily a spinor (as we know, a scalar field satisfies the Klein–Gordon equation, but it is not a spinor). Therefore, we need the Dirac equation to account for the spinor structure. Finally, the fact that a Dirac spinor is also a solution to the Klein–Gordon equation guarantees that the relativistic mass-shell condition is satisfied.

2 Derivation of the Dirac equation from the Dirac action

The Euler–Lagrange equations are,

$$\frac{\partial L}{\partial \phi_\alpha} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial \phi_{\alpha,\mu}}\right) = 0, \tag{17}$$

where the label α runs over the different dynamical fields. In our case, $\phi_1 \equiv \psi$ and $\phi_2 \equiv \overline{\psi}$. We can derive the Dirac equation by using (17), but now we use a different strategy, i.e., we vary the Dirac action with respect to the Dirac field ψ and its conjugate field $\overline{\psi} = \psi^\dagger \gamma^0$ separately.

The Dirac action is

$$S := \int \mathrm{d}^4x\, \overline{\psi}\left(\mathrm{i}\partial\!\!\!/ - m\right)\psi. \tag{18}$$

We obtain the Dirac equation by varying the action with respect to $\overline{\psi}$,

$$0 = \delta S = \int \mathrm{d}^4 x \, \delta\overline{\psi} \left(\mathrm{i}\partial\!\!\!/ - m \right) \psi. \tag{19}$$

Since the variation δS has to be zero for every variation $\delta\overline{\psi}$ of the conjugate Dirac field, and the integrand is continuous, it follows that

$$\left(\mathrm{i}\partial\!\!\!/ - m \right) \psi = 0, \tag{20}$$

i.e., the Dirac equation. We now vary with respect to the Dirac field ψ and obtain

$$\begin{aligned} 0 = \delta S &= \int \mathrm{d}^4 x \, \overline{\psi} \left(\mathrm{i}\partial\!\!\!/ - m \right) \delta\psi \\ &= \int \mathrm{d}^4 x \, \overline{\psi} \left(\mathrm{i}\gamma^\mu \partial_\mu - m \right) \delta\psi, \text{ now we integrate by parts,} \\ &= \int \mathrm{d}^4 x \, \left[-\mathrm{i}\partial_\mu \left(\overline{\psi}\gamma^\mu \right) - \overline{\psi} m \right] \delta\psi, \text{ but the } \gamma \text{ matrices are constant,} \\ &= -\int \mathrm{d}^4 x \, \left(\mathrm{i}\partial\!\!\!/ \overline{\psi} + \overline{\psi} m \right) \delta\psi, \end{aligned} \tag{21}$$

where the boundary term in the integration by parts is zero by the assumptions that the Dirac field and its conjugate approach zero at infinity. We then obtain the adjoint Dirac equation,

$$\left(\mathrm{i}\partial\!\!\!/ + m \right) \overline{\psi} = 0. \tag{22}$$

3 Lorentz covariance of the Dirac equation

We want to show that the Dirac equation is covariant, i.e., invariant in form, under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$. Since the Dirac field does not have any spacetime index on it, we could think that it is a scalar field. However, it is not, as we will see. The Dirac field is a spinor field, and as such it possesses another set of indices (the spinor indices) which are usually not written explicitly when one does not need them.[1] The Dirac field ψ should really be written ψ_α and it is a four-dimensional spinor (each of the 4 γ matrices, $\gamma^0_{\alpha\beta}$, $\gamma^1_{\alpha\beta}$, $\gamma^2_{\alpha\beta}$, $\gamma^3_{\alpha\beta}$, are also 4×4).

We don't know how a spinor transforms under a Lorentz transformation, therefore we make an ansatz. We introduce an invertible *constant* matrix $S(\Lambda)$ which depends on the Lorentz transformation $\Lambda^\mu{}_\nu$, such that,[2]

$$\psi'(x') = S(\Lambda)\psi(x), \quad \text{or} \quad \psi'_\alpha(x') = S(\Lambda)_{\alpha\beta}\psi(x)_\beta, \tag{23}$$

[1] We will encounter situations when it is needed to write down the spinor indices when we will study scattering processes in QED.

[2] Remind that $S(\Lambda)$ is a matrix in spinor space, as the γ matrices.

which of course implies $\psi(x) = S^{-1}(\Lambda)\psi'(x')$.[3] Analogously, we don't know how the γ matrices transform under a Lorentz transformation. They do possess a spacetime index, though, so our first guess would be that they transform as spacetime vectors.

Since we do not know how the γ matrices transform, we cannot compute directly the Dirac equation in the new coordinate system. We have to perform the computation step by step. Let's start by using the definition of $S(\Lambda)$ to rewrite the Dirac equation,

$$\begin{aligned} \left(\mathrm{i}\partial\!\!\!/ - m\right)\psi(x) &= 0 \\ \left(\mathrm{i}\partial\!\!\!/ - m\right)S^{-1}(\Lambda)\psi'(x') &= 0 \\ \mathrm{i}\gamma^\mu S^{-1}(\Lambda)\partial_\mu\psi'(x') - mS^{-1}(\Lambda)\psi'(x') &= 0. \end{aligned} \tag{24}$$

Next, we transform the derivatives,

$$\partial_\mu\psi'(x') = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial\psi'(x')}{\partial x'^\nu} = \Lambda^{\nu'}{}_\mu\partial_{\nu'}\psi'(x') \tag{25}$$

and obtain

$$\mathrm{i}\gamma^\mu S^{-1}(\Lambda)\Lambda^{\nu'}{}_\mu\partial_{\nu'}\psi'(x') - mS^{-1}(\Lambda)\psi'(x') = 0. \tag{26}$$

We can now multiply the expression by $S(\Lambda)$ from the left,[4]

$$\begin{aligned} \mathrm{i}S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^{\nu'}{}_\mu\partial_{\nu'}\psi'(x') - mS(\Lambda)S^{-1}(\Lambda)\psi'(x') &= 0 \\ \mathrm{i}\left[S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^{\nu'}{}_\mu\right]\partial_{\nu'}\psi'(x') - m\psi'(x') &= 0. \end{aligned} \tag{27}$$

In order the Dirac equation to be Lorentz invariant in form (or Lorentz covariant), we need to impose that,

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^{\nu'}{}_\mu \stackrel{!}{=} \gamma^{\nu'} \Longrightarrow S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = \Lambda^\mu{}_{\nu'}\gamma^{\nu'}. \tag{28}$$

This tells us that, under a Lorentz transformation, the γ matrices transform between themselves as the components of a vector, which was our first guess. This equation allows us to determine the expression for $S(\Lambda)$ for infinitesimal Lorentz transformations $\Lambda^\mu{}_{\nu'} = \delta^\mu{}_{\nu'} + \omega^\mu{}_{\nu'}$, with $\omega_{\mu\nu'} = -\omega_{\nu'\mu}$. We start from the following ansatz for $S(\Lambda)$,

$$S\left(\delta^\mu{}_{\nu'} + \omega^\mu{}_{\nu'}\right) = \mathbb{1} + \frac{1}{2}\omega_{\mu\nu'}M^{\mu\nu'}, \quad S^{-1}(\delta^\mu{}_{\nu'} + \omega^\mu{}_{\nu'}) = \mathbb{1} - \frac{1}{2}\omega_{\mu\nu'}M^{\mu\nu'}, \tag{29}$$

[3] Note that, unfortunately, it is not common to care about the position of spinor indices (up or down). This means that, in (23), the index β is dummy and contracted even if it appears twice in covariant position, whereas the α index is free. *The position of spacetime indices remains fundamental, though!*

[4] Since $S(\Lambda)$ is invertible, we do not lose any information when we apply it to (26).

valid up to the linear order in $\omega_{\mu\nu'}$.[5] We now insert this ansatz in (28), remove the primes for simplicity (all primed indices are contracted in the equation, so we can rename them freely), and solve for $M^{\mu\nu}$ keeping only the terms up to linear order in $\omega_{\mu\nu}$,

$$\left[\mathbb{1} + \frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right]\gamma^\mu\left[\mathbb{1} - \frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right] = \left[\delta^\mu{}_\nu + \omega^\mu{}_\nu\right]\gamma^\nu, \tag{30a}$$

$$\gamma^\mu - \frac{1}{2}\gamma^\mu\omega_{\rho\sigma}M^{\rho\sigma} + \frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\gamma^\mu = \gamma^\mu + \omega^\mu{}_\nu\gamma^\nu, \tag{30b}$$

$$\frac{1}{2}\omega_{\rho\sigma}\left(M^{\rho\sigma}\gamma^\mu - \gamma^\mu M^{\rho\sigma}\right) = \omega^\mu{}_\nu\gamma^\nu. \tag{30c}$$

Since $\omega_{\mu\nu}$ is antisymmetric, we can write

$$\omega^\mu{}_\nu = \eta^{\mu\rho}\omega_{\rho\nu} = \eta^{\mu\rho}\frac{1}{2}\left(\omega_{\rho\nu} - \omega_{\nu\rho}\right) = \frac{1}{2}\left(\omega^\mu{}_\nu - \omega_\nu{}^\mu\right). \tag{31}$$

It follows

$$\frac{1}{2}\omega_{\rho\sigma}\left(M^{\rho\sigma}\gamma^\mu - \gamma^\mu M^{\rho\sigma}\right) = \frac{1}{2}\left(\omega^\mu{}_\nu - \omega_\nu{}^\mu\right)\gamma^\nu, \tag{32a}$$

$$\omega_{\rho\sigma}\left(M^{\rho\sigma}\gamma^\mu - \gamma^\mu M^{\rho\sigma}\right) = \omega_{\rho\sigma}\left(\eta^{\rho\mu}\delta^\sigma{}_\nu - \delta^\rho{}_\nu\eta^{\sigma\mu}\right)\gamma^\nu, \tag{32b}$$

$$M^{\rho\sigma}\gamma^\mu - \gamma^\mu M^{\rho\sigma} = \left(\eta^{\rho\mu}\delta^\sigma{}_\nu - \delta^\rho{}_\nu\eta^{\sigma\mu}\right)\gamma^\nu. \tag{32c}$$

This last equation can be solved for $M^{\rho\sigma}$ and the solution is

$$M^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right]. \tag{33}$$

With the explicit expression for $M^{\mu\nu}$ in hand, it can be shown that $S^\dagger(\Lambda) = \gamma^0 S^{-1}(\Lambda)\gamma^0$ (please check it). We define a "spinor" as an object which transform as follows under an infinitesimal Lorentz transformation (compare with (A.9) and (A.60) in [M&S], where $\omega_{\mu\nu}$ is called $\epsilon_{\mu\nu}$),

$$\begin{aligned}\psi'(x') &= \left(\mathbb{1} + \frac{1}{8}\omega_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right]\right)\psi(x) = \left[\mathbb{1} - \frac{\mathrm{i}}{4}\omega_{\mu\nu}\left(\frac{\mathrm{i}}{2}\left[\gamma^\mu, \gamma^\nu\right]\right)\right]\psi(x) \\ &= \left(\mathbb{1} - \frac{\mathrm{i}}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi(x)\end{aligned} \tag{34}$$

Demanding the Lorentz covariance of the Dirac equation is powerful enough to determine $S(\Lambda)$, i.e., to determine how the γ matrices and the Dirac field ψ transform under a Lorentz transformation.

Exercise 2. Check whether $\psi^\dagger(x)\psi(x)$ and $\overline{\psi}(x)\psi(x)$ are Lorentz covariant.

[5] Note that the identity in (29) is the identity in the spinor space, and $M^{\mu\nu'}$ also has spinor indices.

Solution. Under a Lorentz transformation, the first object transforms as $\psi^\dagger(x)\psi(x) \longrightarrow \psi^\dagger(x')S^\dagger(\Lambda)S(\Lambda)\psi(x')$, which is not invariant in form because $S^\dagger(\Lambda)S(\Lambda)$ is not the identity, i.e., $S(\Lambda)$ is not unitary. This means that $\psi^\dagger(x)\psi(x)$ is not a scalar field.

The second object transforms as,

$$\begin{aligned}\overline{\psi}(x)\psi(x) \longrightarrow \quad & \psi^\dagger(x')S^\dagger\gamma^0 S\psi(x') \\ & = \psi^\dagger(x')\left[\gamma^0 S^{-1}\gamma^0\right]\gamma^0 S\psi(x'), \text{ but } \gamma^0\gamma^0 = \mathbb{1}, \\ & = \psi^\dagger(x')\gamma^0 S^{-1}S\psi(x') \\ & = \psi^\dagger(x')\gamma^0\psi(x') \\ & = \overline{\psi}(x')\psi(x'). \end{aligned} \tag{35}$$

Therefore, $\overline{\psi}(x)\psi(x)$ is invariant in form and it is a scalar field. We will see that both quantities are used when studying the solution to the Dirac equation.

4 The solution to the Dirac equation

We are going to solve the Dirac equation for a free particle. Suppose that the solution has the form of a plane wave,

$$\psi(x) = u\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px}, \text{ with } u\left(\vec{p}\right) \text{ four-component spinor.} \tag{36}$$

The Dirac equation becomes,

$$\begin{aligned} 0 &= \left(\mathrm{i}\not{\partial} - m\right)\psi(x) = \left(\mathrm{i}\not{\partial} - m\right)u\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px} \\ &= \mathrm{i}\gamma^\mu\partial_\mu\left(u\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px}\right) - mu\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px} \\ &= \mathrm{i}\gamma^\mu\left(-p_\mu\right)u\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px} - mu\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px} \\ &= \left(\gamma^\mu p_\mu - m\right)\psi(x) = \left(\not{p} - m\right)\psi(x). \end{aligned} \tag{37}$$

It follows,

$$\left(\not{p} - m\right)u\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px} = 0 \Longrightarrow \left(\not{p} - m\right)u\left(\vec{p}\right) = 0. \tag{38}$$

This is equivalent to taking the Fourier transform of the Dirac equation. Let's solve (38); we have,

$$\left(p_0\gamma^0 + p_i\gamma^i - m\right)u\left(\vec{p}\right) = 0, \tag{39}$$

with

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \tag{40}$$

where the σ_k are the Pauli matrices. Given the 2-block structure of the γ matrices, we split the four-component Dirac spinor into two two-components spinors φ and χ,

$$u\left(\vec{p}\right) = \begin{pmatrix} \varphi \\ \tilde{\chi} \end{pmatrix}. \tag{41}$$

The Dirac equation then reads,

$$\begin{pmatrix} p_0 - m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -p_0 - m \end{pmatrix} \begin{pmatrix} \varphi \\ \tilde{\chi} \end{pmatrix} = 0, \tag{42}$$

which can be rewritten as the following system

$$\left(p_0 - m\right)\varphi - \left(\vec{p}\cdot\vec{\sigma}\right)\tilde{\chi} = 0, \tag{43a}$$

$$\left(\vec{p}\cdot\vec{\sigma}\right)\varphi - \left(p_0 + m\right)\tilde{\chi} = 0. \tag{43b}$$

From the second equation we get $\tilde{\chi} = \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0 + m}\varphi$, and the substitution in the first equation gives,

$$0 = \left(p_0 - m\right)\varphi - \left(\vec{p}\cdot\vec{\sigma}\right)\frac{\vec{\sigma}\cdot\vec{p}}{p_0 + m}\varphi = \left(p_0^2 - m^2\right)\varphi - \left(\vec{p}\cdot\vec{\sigma}\right)^2\varphi. \tag{44}$$

I leave as an exercise for you to show that $\left(\vec{p}\cdot\vec{\sigma}\right)^2 = \left(p_1^2 + p_2^2 + p_3^2\right)\mathbb{1}$. It follows that,

$$0 = \left(p_0^2 - m^2\right)\varphi - \left(p_1^2 + p_2^2 + p_3^2\right)\mathbb{1}\varphi = \left(p_0 - |\vec{p}|^2 - m^2\right)\varphi. \tag{45}$$

This tells us that the plane wave $\psi(x) = u\left(\vec{p}\right)\mathrm{e}^{-\mathrm{i}px}$ is a solution to the Dirac equation if the relativistic mass-shell condition is satisfied, which is exactly what we want.

Now we consider the negative energy spinors, $\psi(x) = v\left(\vec{p}\right)\mathrm{e}^{\mathrm{i}px} = \begin{pmatrix} \tilde{\varphi} \\ \chi \end{pmatrix}\mathrm{e}^{\mathrm{i}px}$. The computation is analogous to the one done so far, therefore we will skip some steps now (which you can complete on your own).

$$\left(\mathrm{i}\not{\partial} - m\right)v\left(\vec{p}\right)\mathrm{e}^{\mathrm{i}px} \Longrightarrow \left(\not{p} + m\right)v\left(\vec{p}\right) = 0, \tag{46}$$

with a sign difference in front of the mass with respect to the positive energy spinors. Writing this equation in 2×2 blocks results in,

$$\tilde{\varphi} = -\frac{\vec{\sigma}\cdot\vec{p}}{p_0 + m}\chi \tag{47a}$$

$$0 = \left[p_0^2 - m^2 - \left(\vec{\sigma}\cdot\vec{p}\right)^2\right]\chi \Longrightarrow p_0^2 = |\vec{p}|^2 + m^2. \tag{47b}$$

We get the same condition as for the positive energy spinors.

The two independent Dirac spinors are,

$$u(\vec{p}) = \begin{pmatrix} \varphi \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi \end{pmatrix}, \qquad v(\vec{p}) = \begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi \\ \chi \end{pmatrix}. \tag{48}$$

Both φ and χ are not determined. However, they are two 2-spinors, so they can be expressed in terms of the canonical basis in the spinor space,

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{49}$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{50}$$

Since the Dirac equation is linear, all its solutions can be written in terms of this canonical basis, so we can consider it as our solutions without losing generality. The physical interpretation of this is that, for both particles and antiparticles $[u(\vec{p})$ and $v(\vec{p})]$ we have spin up (φ_1 and χ_1) and spin down (φ_2 and χ_2), which is what we need to describe spin-1/2 fermions.

The two independent solutions to the Dirac equations are,

$$\psi_r^+(x) \propto \begin{pmatrix} \varphi_r \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r \end{pmatrix} \mathrm{e}^{-\mathrm{i}px}, \quad \psi_r^-(x) \propto \begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_r \\ \chi_r \end{pmatrix} \mathrm{e}^{\mathrm{i}px}, \tag{51}$$

where the superscripts $\pm$ indicate the positive or negative energies, and the index r runs over $1, 2$.

Now we need to determine the normalization for these solutions. Let's consider the positive energy spinors $u_r(\vec{p})$ first. We consider two possibilities: the "covariant normalization" $\overline{u}_r(\vec{p})\,u_r(\vec{p}) = 1$ and the "Dirac normalization" $u_r^\dagger(\vec{p})\,u_r(\vec{p}) = 1$. These two normalizations have both a pro and a con; the covariant normalization is covariant (hence the name) as we saw explicitly in the previous section, but $\overline{\psi}\psi$ is not positive definite. The Dirac normalization, instead, is not covariant, but $\psi^\dagger\psi$ is positive definite. Let's start with the covariant normalization.

$$\begin{aligned}
\overline{u}_r(\vec{p})\,u_r(\vec{p}) = u_r^\dagger(\vec{p})\,\gamma^0 u_r(\vec{p}) &= \begin{pmatrix} \varphi_r^\dagger & \varphi_r^\dagger\dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \varphi_r \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r \end{pmatrix} \\
&= \begin{pmatrix} \varphi_r^\dagger & \varphi_r^\dagger\dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m} \end{pmatrix} \begin{pmatrix} \varphi_r \\ -\dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r \end{pmatrix} = \varphi_r^\dagger\varphi_r - \varphi_r^\dagger\frac{(\vec{\sigma}\cdot\vec{p})^2}{(p_0+m)^2}\varphi_r \\
&= \frac{1}{(E+m)^2}\left((E+m)^2 - |\vec{p}|^2\right)\varphi_r^\dagger\varphi_r.
\end{aligned} \tag{52}$$

Since the φ_r are the canonical vectors, their norm is 1. We get,

$$\begin{aligned}\overline{u}_r(\vec{p})\,u_r(\vec{p}) &= \frac{E^2+m^2+2mE-|\vec{p}|^2}{(E+m)^2} \\ &= \frac{|\vec{p}|^2+m^2+m^2+2mE-|\vec{p}|^2}{(E+m)^2} = \frac{2m}{(E+m)}.\end{aligned} \tag{53}$$

Therefore, the covariantly normalized solution is

$$\psi_r^+(x) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\varphi_r \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r\end{pmatrix}\mathrm{e}^{-\mathrm{i}px}. \tag{54}$$

For the negative energy spinors we need to impose $\overline{v}_r(\vec{p})\,v_r(\vec{p}) = -1$, which results in (please check it)

$$\psi_r^-(x) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}-\dfrac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_r \\ \chi_r\end{pmatrix}\mathrm{e}^{\mathrm{i}px}. \tag{55}$$

From now on, $u_r(\vec{p})$ and $v_r(\vec{p})$ will denote the normalized spinors. Now, the Dirac equation and its adjoint tell us that

$$\not{p}v_r(\vec{p}) = -mv_r(\vec{p}), \quad \not{p}u_r(\vec{p}) = mu_r(\vec{p}), \tag{56}$$

i.e., $v_r(\vec{p})$ and $u_r(\vec{p})$ are eigenvectors of $\not{p}$ with different eigenvalues. Then, by the (finite-dimensional) spectral theorem, if $\not{p}$ is hermitian with respect to the covariant inner product (i.e., with respect to the covariant normalization), $v_r(\vec{p})$ and $u_r(\vec{p})$ will be orthogonal. We can show explicitly that $\not{p}$ is hermitian with respect to the covariant inner product. Suppose we have two solutions to the Dirac equation ψ_1 and ψ_2; then

$$\begin{aligned}(\psi_2,\not{p}\psi_1) &= \overline{\psi}_2(\not{p}\psi_1) = \psi_2^\dagger\gamma^0(\not{p}\psi_1) = \psi_2^\dagger\gamma^0(m\psi_1) = \left(m\psi_2^\dagger\right)\gamma^0\psi_1 \\ &= \left(\psi_2^\dagger\not{p}^\dagger\right)\gamma^0\psi_1 = (\not{p}\psi_2)^\dagger\gamma^0\psi_1 = \overline{(\not{p}\psi_2)}\psi_1 = (\not{p}\psi_2,\psi_1).\end{aligned} \tag{57}$$

It follows that $v_r(\vec{p})$ and $u_r(\vec{p})$ are orthogonal and we can define projectors on the positive energy states and negative energy states, by means of the completeness relation,

$$\mathbb{1} = \sum_r\left[u_r(\vec{p})\,\overline{u}_r(\vec{p}) - v_r(\vec{p})\,\overline{v}_r(\vec{p})\right]. \tag{58}$$

The two projectors are,

$$\Lambda_+(\vec{p}) \coloneqq \sum_r u_r(\vec{p})\,\overline{u}_r(\vec{p}), \quad \Lambda_-(\vec{p}) \coloneqq \sum_r v_r(\vec{p})\,\overline{v}_r(\vec{p}), \tag{59}$$

with $\Lambda_+(\vec{p}) + \Lambda_-(\vec{p}) = \mathbb{1}$. I leave you as an exercise to show that

$$\Lambda_+(\vec{p}) = \frac{\not{p} + m}{2m}, \quad \Lambda_-(\vec{p}) = \frac{m - \not{p}}{2m}. \tag{60}$$

Here we show how these projectors act on the basis spinors $u_r(\vec{p})$ and $v_r(\vec{p})$, which is sufficient to determine their action over the entire vector space of spinors,

$$\Lambda_+(\vec{p})\, u_r(\vec{p}) = \frac{\not{p} + m}{2m} u_r(\vec{p}) = \frac{m + m}{2m} u_r(\vec{p}) = u_r(\vec{p}), \tag{61a}$$

$$\Lambda_+(\vec{p})\, v_r(\vec{p}) = \frac{\not{p} + m}{2m} v_r(\vec{p}) = \frac{m - m}{2m} u_r(\vec{p}) = 0, \tag{61b}$$

$$\Lambda_-(\vec{p})\, u_r(\vec{p}) = \frac{m - \not{p}}{2m} u_r(\vec{p}) = \frac{m - m}{2m} u_r(\vec{p}) = 0, \tag{61c}$$

$$\Lambda_-(\vec{p})\, v_r(\vec{p}) = \frac{m - \not{p}}{2m} v_r(\vec{p}) = \frac{m + m}{2m} u_r(\vec{p}) = v_r(\vec{p}). \tag{61d}$$

where we used the Dirac equation and its adjoint.

We now compute the Dirac normalization, starting from the covariant one.

$$\begin{aligned} u_r^\dagger(\vec{p})\, u_r(\vec{p}) &= \frac{E+m}{2m} \begin{pmatrix} \varphi_r^\dagger & \varphi_r^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m} \end{pmatrix} \begin{pmatrix} \varphi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r \end{pmatrix} \\ &= \frac{E+m}{2m} \left(\varphi_r^\dagger \varphi_r + \varphi_r^\dagger \frac{(\vec{\sigma}\cdot\vec{p})^2}{(p_0+m)^2} \varphi_r \right) \\ &= \frac{E+m}{2m} \frac{E^2 + m^2 + 2mE + |\vec{p}|^2}{(E+m)^2} = \frac{1}{2m} \frac{2E^2 + 2mE}{E+m} = \frac{E}{m}. \end{aligned} \tag{62}$$

The same result holds for $v_r^\dagger(\vec{p})\, v_r(\vec{p})$. Hence the Dirac spinors are,

$$\begin{aligned} \tilde{u}_r(\vec{p}) &= \sqrt{\frac{m}{E}} u_r(\vec{p}) = \sqrt{\frac{m}{E}} \sqrt{\frac{E+m}{2m}} \begin{pmatrix} \varphi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r \end{pmatrix} \\ &= \sqrt{\frac{E+m}{2E}} \begin{pmatrix} \varphi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\varphi_r \end{pmatrix}, \end{aligned} \tag{63a}$$

$$\begin{aligned} \tilde{v}_r(\vec{p}) &= \sqrt{\frac{m}{E}} v_r(\vec{p}) = \sqrt{\frac{m}{E}} \sqrt{\frac{E+m}{2m}} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_r \\ \chi_r \end{pmatrix} \\ &= \sqrt{\frac{E+m}{2E}} \begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{p_0+m}\chi_r \\ \chi_r \end{pmatrix}. \end{aligned} \tag{63b}$$

Finally, we can write down the general solution of the Dirac equation, normalized à la Dirac, thanks to the fact that it is a linear equation and therefore we can apply the superposition principle,

$$\psi(x) = \sum_{r=1}^{2} \int \frac{\mathrm{d}\vec{p}}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \left[c_r(\vec{p})\, u_r(\vec{p})\, \mathrm{e}^{-\mathrm{i}px} + d_r^*(\vec{p})\, v_r(\vec{p})\, \mathrm{e}^{\mathrm{i}px} \right]. \tag{64}$$

If one wants to consider the covariant normalization, it suffices to remove the factor $\sqrt{\frac{m}{E}}$. Upon quantization, the complex coefficients $c_r(\vec{p})$ and $d_r^*(\vec{p})$ become operators.